Correlation parameterization and calibration for the LIBOR market model

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Chapter 1

Introduction

Interest rate products and interest rate derivatives are one of the largest areas in financial markets. Historically, interest rate models were based on the short rate or on instantaneous forward rates as stochastic elements. In the late nineties a new class of models, called market models, was introduced by (Brace et al., 1997), (Miltersen et al., 1997) and (Jamshidian, 1997). The distinguishing feature of these models is their ability to exactly reproduce prices that are observed in the market. In particular, the LIBOR market model (LIBOR, Libor Interbank Offered Rate) is able to reproduce market-quoted implied cap volatilities, and the swap rate model is able to reproduce market-quoted swaption volatilities.

In the LIBOR market model and the swap rate model, the underlying interest rates are modelled by a number of correlated stochastic processes, also called driving factors. Once the model has been calibrated, i.e. adjusted to market-observed rates, products for which a closed pricing formula does not exist, can be priced by running Monte Carlo simulation on the model. Examples of such products are various cap-based products, such as the ratchet cap, the sticky cap and the flexi cap[1], or Bermudan swaptions.

Crucial inputs to the model are the volatility structure of the interest rates and the correlation of the stochastic processes that drive the model. Intuitively, it is clear that two interest rates with maturities that lie close to each other move more in-line with each other than interest rates whose maturities lie far apart from each other.

[1]The cap rate of a ratchet cap equals the LIBOR rate of the previous reset date plus a spread. The sticky cap's cap rate is the previous capped rate plus a spread, and the flexi cap has a limit on the total number of caplets that can be exercised.

In contrast, one-factor models, such as the model by (Ho and Lee, 1986), where the short rate is modelled by one stochastic process, assume that the resulting term structure is perfectly correlated for all maturities.
Several approaches on the estimation and representation of the volatility and correlation structures have been proposed since the emergence of market models. There are two common approaches for obtaining the correlation matrix: it is either estimated from historical data, such as forward rates, or it is retrieved and bootstrapped directly from correlation-sensitive market-quoted instruments such as European swaptions. Fitting such an estimated correlation matrix to a parameterized functional form has several advantages. First, it is ensured that the correlation matrix fulfills general properties that have been observed in the market, specifically for interest rate correlation. An example for such a property is the observation that correlation among adjacent interest rates is high and tends to decrease with increasing maturity interval. This property is enforced by a functional form, even if the concrete data that is used does not, by chance, fulfil this property. Second, fitting the data to a parameterized functional form allows for controlling the rank of the correlation matrix. By coincidence, it can happen that the estimated correlation matrix does not have full rank, which reduces the number of driving factors in the model. A full-rank parameterization guarantees, as its name implies, a full-rank correlation matrix. On the other hand, it may be desirable to use a model with fewer driving factors as rates in the model, in which case a reduced-rank parameterization can be applied.
The objective of this thesis is to present the standard approaches for parameterizing the correlation matrix that have been established over the past few years. Forward rates being the underlying of caplets, it is analysed how correlation of forward rates of different tenors are linked. Chapter 2 introduces the necessary notation for interest rates and interest rate products used within this thesis, and the LIBOR market model is presented. Given historical money market and swap rates, forward rates of 3-month tenor and 6-month tenor were calculated and their correlation was estimated. This is outlined in detail in chapter 3. Chapter 4 then presents some full-rank and reduced-rank parameterizations for fitting the historical market correlation. The parameters of the full-rank parameterizations are analysed in detail. The focus of this thesis is on full-rank parameterizations, but for completeness some popular reduced-rank paramterizations are introduced. How correlation for forward rates of different tenors is associated is analysed in chapter 5. Detailed results of all parameterizations are

given in chapter 6. Chapter 7 contains a review of this work and some concluding remarks. A brief overview of concepts of stochastic calculus that are required for understanding the description of the LIBOR market model is given in appendix A. Appendix B contains some calculations of the derivation of one parameterization that was incorrectly referred to in one reference.

Based on an existing implementation from HfB, the parameterizations were applied to historical market data. Some parameterizations presented here were added to the existing implementation. The implementation was done in *C#*.

The main references used for this thesis are some of the standard references on interest rate modelling, namely (Brigo and Mercurio, 2001), (Rebonato, 2002) and (Rebonato, 2004), and current papers that address the parameterization of correlation for the LIBOR market model, namely (Brigo, 2002), (Schoenmakers and Coffey, 2000), (Alexander, 2003) and (Alexander and Lvov, 2003).

Chapter 2

Interest rate products and the LIBOR Market Model

Some basic definitions that will be used throughout the thesis are required and sketched below. These definitions can be found in any of the standard text books on financial derivatives or interest-rate modelling, such as (Baxter and Rennie, 1998), (Pelsser, 2000), (Brigo and Mercurio, 2001), (Rebonato, 2002), (Joshi, 2003) and (Hunt and Kennedy, 2004).

2.1 Interest rates

An interest rate denotes the time value of money. It is the rate at which an amount of money accrues over time. The worth of €1 today is not the same as that of €1 one year later.
As interest rates are not constant over time, the common approach of interest rate modelling is to describe interest rates as stochastic processes that evolve over time.
An interest rate model can then be used to value and price financial securities that are based on interest rates and that cannot be valued by a replication strategy. Typically, when no closed formula can be obtained for the valuation of a financial product, Monte Carlo simulation is applied.
One definition of an interest rate is that of the *short rate* or *instantaneous interest rate*, defined by the *Bank account*:

Definition 2.1 (Bank account). Let $B(t)$ be the value of a bank account at time $t \geq 0$, and $B(0) = 1$. The bank account evolves over time according to the differential equation

$$dB(t) = r_t B(t) dt,$$

where r_t is a positive function of time. A solution for the differential equation is

$$B(t) = \exp\left(\int_0^t r_s ds\right). \tag{2.1}$$

The short rate denotes the interest rate at which money accrues when being continuously re-invested.
The amount to invest at time t to yield 1 at time T is given by the *stochastic discount factor*:

Definition 2.2 (Stochastic discount factor). The (stochastic) discount factor $D(t,T)$ denotes the amount at time t that is equivalent to one unit at time T, given by

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s ds\right).$$

The stochastic discount factor $D(t,T)$ is reflected in the financial market by the *zero-coupon bond*, also referred to as *pure discount bond*:

Definition 2.3 (Zero-coupon bond, (deterministic) discount factor). A zero-coupon bond with maturity T guarantees the holder the payment of one unit of currency at maturity. The value of the contract at time t is denoted by $P(t,T)$. By definition, it holds that $P(T,T) = 1$.

Whereas a zero-coupon bond has a price at each time t, the stochastic discount factor denotes a stochastic process. Under a suitable probability measure, $P(t,T)$ can be viewed as the expectation of $D(t,T)$.
Other interest rate compounding conventions exist that entail a day-count convention:

Definition 2.4 (Day-count convention). A *day-count convention* $\tau(t,T)$ denotes the number of days between two points in time t and T. Depending on the type of product to be valued, different *day-count conventions* for interest-rate compounding apply. Some common day-count conventions are:

- $\frac{\text{Act}}{360}$,
- $\frac{\text{Act}}{365}$,

- $\frac{30}{360}$.

There are different interest rate compounding conventions:

Definition 2.5 (Interest rate compounding). (i) The *continuously-compounded spot interest rate* denotes the constant rate at which an amount accrues continuously for an investment at time t with maturity T:

$$R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)}. \tag{2.2}$$

(ii) The *simply-compounded spot interest rate* is the constant rate at which an investment has to be made to produce an amount of one unit at maturity T, starting from $P(t,T)$ units at time t:

$$L(t,T) = \frac{1}{\tau(t,T)}\left(\frac{1}{P(t,T)} - 1\right). \tag{2.3}$$

Examples for simply-compounded rate are the LIBOR rate (London Interbank Offered Rate) and the EURIBOR rate (Euro Interbank Offered Rate), which denote interest rates at which deposits are exchanged within banks. LIBOR rates for various currencies (including the Euro) and tenors are fixed daily by the British Bankers' Association. They are averages derived from quotations provided by banks. Similarly, EURIBOR rates are fixed daily by the European Banking Federation (FBE) from average quotes of several banks. All references to LIBOR rates in this text are equally valid for EURIBOR rates.

(iii) An investment starting at time T_1 with maturity T_2 is simply-compounded at time $t < T_1$ with the *simply-compounded forward interest rate* denoted by

$$F(t;T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left(\frac{P(t,T_1)}{P(t,T_2)} - 1\right). \tag{2.4}$$

It is the fair rate of a forward rate agreement (FRA) with expiry T_1 and maturity T_2 at time t. Under the risk-neutral probability measure of the discount bond $P(t,T_2)$ the forward rate $F(t;T_1,T_2)$ is the expectation of the LIBOR rate $L(T_1,T_2)$ at time t. It is therefore often called the *forward LIBOR rate* or just the *LIBOR rate*.

(iv) The *discretely-compounded interest rate* is the constant rate at which an amount invested at time t accrues if proceeds are re-invested k times per year:

$$Y^k(t,T) = \frac{k}{P(t,T)^{\frac{1}{k\tau(t,T)}}} - k.$$

The special case where $k = 1$ is called *annual compounding*.

Just as there is an instantaneous interest rate, there is also an *instantaneous forward rate*:

Definition 2.6 (Instantaneous forward rate). Borrowing at some point in time T with instantaneous maturity T_2, i.e. $\lim_{T_2 \to T}$, yields the instantaneous forward rate:

$$f(t,T) = \lim_{T_2 \to T} F(t;T,T_2) = -\frac{\partial}{\partial T} \ln P(t,T). \tag{2.5}$$

2.2 Interest rate derivatives

This section introduces those interest rate products that will be used in this text. These are interest rate swaps, forward start swaps, caps and floors, and swaptions.

2.2.1 Interest rate swaps

One of the most liquid interest rate derivatives is the interest rate swap, often referred to simply as swap when obvious from the context. It is an exchange of a stream of fixed payments for a stream of floating (varying) payments. The *fixed leg* denotes a stream of annual fixed payments, and its rate is specified on inception of the contract. The day-count convention for the fixed payments is typically $\frac{30}{360}$. The *floating leg* consists of a stream of varying payments. If payment dates are $T_i = T_0 + i\delta$, $i = 1, \ldots, n$, then the ith payment will be fixed at T_{i-1} to be the LIBOR or EURIBOR rate $L(T_{i-1}, T_i)$. The floating payments are usually of semi-annual tenor. A swap where the holder receives fixed payments and pays variable payments is called a *receiver swap*. It is equal to a portfolio which is long a fixed coupon bond and short a floating coupon bond. The opposite position is called *payer swap*.

The *par swap rate* or just *swap rate* is the fair coupon rate of a swap, i.e. it is the rate of fixed coupon payments which makes the value of the swap 0.

The value of the floating leg is a series of forward rate agreements, where the payment at time T_i is based on the forward rate $F(t;T_{i-1},T_i)$ from equation (2.4). Assuming a notional of 1, the forward rate can be re-written to denote the floating payment:

$$\begin{aligned} \tilde{V}_i(t) &= F(t,T_{i-1},T_i)\tau(T_{i-1},T_i)P(t,T_i) \\ &= P(t,T_{i-1}) - P(t,T_i). \end{aligned}$$

The value of the floating leg with payments at T_i, $i = 1, \dots, n$, is the sum of the individual payments:

$$\begin{aligned}\tilde{V}(t) &= \sum_{i=n}^{N} P(t, T_{i-1}) - P(t, T_i) \\ &= P(t, T_{n-1}) - P(t, T_N).\end{aligned}$$

The value of the fixed leg is given by the discounted coupon payments:

$$\bar{V}(t) = c \cdot \sum_{i=n}^{N} P(t, T_i)\tau(T_{i-1}, T_i).$$

The value of a receiver swap is then given by

$$V(t) = \bar{V}(t) - \tilde{V}(t),$$

and the swap rate is the rate c that yields $V(t) = 0$, i.e.

$$c(t, T_n, T_N) := \frac{P(t, T_{n-1}) - P(t, T_N)}{\sum_{i=n}^{N} P(t, T_i)\tau(T_{i-1}, T_i)}, \quad T_n \geq t. \tag{2.6}$$

The denominator is called the *present value of a basis point* or *PVBP*.
In the case where the swap rate denotes the spot swap rate, i.e. $T_{n-1} = t$, this reduces to

$$c(t, T_N) := \frac{1 - P(t, T_n)}{\sum_{i=1}^{N} P(t, T_i)\tau(T_{i-1}, T_i)}. \tag{2.7}$$

For $T_{n-1} > t$, the swap rate denotes the *forward par swap rate*, the rate at which a forward start swap, or forward swap, is initiated. This is an agreement to enter into a swap at a future point in time with the details of the contract being specified at t.

2.2.2 Caps and floors

A cap is a series of options that insure against rising interest rates. There is one option per period, a *caplet*, that, in case of option exercise, pays the difference between an interest rate, e.g. LIBOR, and the cap rate k at time T_i:

$$C_{i-1}(T_i) = \tau(T_{i-1}, T_i)(L(t, T_{i-1}, T_i) - k)^+,$$

where $L(t, T_{i-1}, T_i)$ denotes the forward LIBOR rate between T_{i-1} and T_i at time t, and $(x)^+ := \max(x, 0)$. A caplet is thus a call option on a forward rate, and a cap is a series of such call options.

Using equation (2.4), the forward LIBOR rate is given by

$$\tau(T_{i-1}, T_i)L(t, T_{i-1}, T_i) = \frac{P(t, T_{i-1}) - P(t, T_i)}{P(t, T_i)}.$$

Using the discount bond $P(t, T_i)$ as the numeraire, the forward LIBOR rate will be a martingale under the measure $\mathbb{Q}^i$ associated with $P(t, T_i)$. It is assumed to follow a process given by the SDE

$$dL(t, T_{i-1}, T_i) = \sigma_{i-1}(t)L(t, T_{i-1}, T_i)dW^i,$$

where W^i denotes a Brownian motion under the measure $\mathbb{Q}^i$. The solution is a Geometric Brownian motion, therefore the forward LIBOR rate is assumed to be lognormally distributed. The measure $\mathbb{Q}^i$ is called the *forward measure*.
Given that $\frac{C_{i-1}(t)}{P(t,T_i)}$ is a martingale under the $\mathbb{Q}^i$ measure, it holds that

$$\begin{aligned}
\frac{C_{i-1}(t)}{P(t, T_i)(t)} &= \mathbb{E}^i\left(\frac{C_{i-1}(T_i)}{P(T_i, T_i)}\right) \\
&= \tau(T_{i-1}, T_i)\mathbb{E}^i((L(t, T_{i-1}, T_i) - k)^+),
\end{aligned}$$

where $\mathbb{E}^i$ denotes the expectation under $\mathbb{Q}^i$. Setting $L := L(t, T_{i-1}, T_i)$, this can now be expressed explicitly using (Black, 1976)'s formula for option pricing:

$$C(T_{i-1})(k, L(t, T_{i-1}, T_i), \sigma) = L \cdot N(d_1(k, F, \sigma)) - k \cdot N(d_2(k, L, \sigma)), \tag{2.8}$$

where

$$\begin{aligned}
d_1(k, L, \sigma) &= \frac{\ln\left(\frac{L}{k}\right) + \frac{\sigma^2 \cdot (T_i - T_{i-1})}{2}}{\sigma \cdot \sqrt{T_i - T_{i-1}}}, \\
d_2(k, L, \sigma) &= \frac{\ln\left(\frac{L}{k}\right) - \frac{\sigma^2 \cdot (T_i - T_{i-1})}{2}}{\sigma \cdot \sqrt{T_i - T_{i-1}}}.
\end{aligned}$$

$N(x)$ denotes the standard normal cumulative distribution function. The value of a cap is the sum of the caplet values. It is market convention to quote cap volatilities, as there is a unique relationship between cap prices and cap volatilities using the Black formula. Caplet volatilities are not quoted in the market, but can be retrieved by bootstrapping the market-quoted cap rates.
A floor is a series of put options on forward rates, called *floorlets*, thereby insuring against falling interest rates. The payoff of each floorlet at time T_i in case of exercise is:

$$\tau(T_{i-1}, T_i)(k - L(T_{i-1}, T_i))^+.$$

Buying a floor and selling a cap at the same rate k is equivalent to owning a receiver swap with fixed rate k. This is the *put-call parity* relationship of caps and floors.

2.2.3 Swaptions

A European swaption is an option to enter into a swap on a future date at a given rate. A payer swaption, namely the right to pay a fixed rate, is a call on a forward swap rate. Similar to the argumentation for caplets, the expectation of a swaption becomes a martingale under the swap rate measure, and can then be priced with the Black formula, assuming a lognormal distribution of the swap rate. The swap rate measure is the measure associated with using the PVBP from equation (2.6), $\sum_{i=n}^{N} P(t, T_i)\tau(T_{i-1}, T_i)$, as numeraire.
Forward rates and swap rates cannot be assumed to be lognormal at the same time, as the forward measure cannot be linearly expressed in terms of the swap rate measure, and vice versa. However, the inconsistencies are small and it is market practice to ignore them (see (Joshi, 2003) or (Rebonato, 1999a)).

2.3 Term structure of interest rates

Basic input for valuing interest-rate products is an interest rate curve, also called *term structure of interest rates.* An interest rate curve is a function of maturity to interest rate. Rates up to one year are simply-compounded money market rates, such as LIBOR rates. Interest rates with maturity above one year are typically either one of annually-compounded yield rates, zero-coupon rates, forward rates or swap rates. Due to the liquidity of interest rate swaps, the *swap curve* is a popular interest rate curve. It is defined by

$$T \mapsto \begin{cases} L(t,T) & t \leq t+1 \text{ (years)}, \\ r(t,T) & \text{else}, \end{cases} \tag{2.9}$$

where $r(t, T)$ is the par swap rate given by equation (2.7).

2.4 The LIBOR market model

The class of *market models* was devised in the late 1990s. It was introduced by Brace, Gatarek and Musiela (Brace et al., 1997), Miltersen, Sandmann and Sondermann (Miltersen et al., 1997) for LIBOR-based models and by Jamshidian (Jamshidian, 1997) for swap-based models. The LIBOR market model is also often referred to as BGM (in recognition of Brace, Gatarek and Musiela) or the lognormal forward-LIBOR model (LFM). Comparing the LIBOR market model to existing short-rate models, the new and prominent feature of the model is its calibration to forward LIBOR market rates, namely either cap volatilities or European swaption volatilities, as quoted in the market by the use of Black's formula (see equation 2.8). Once set up, the model can be used to price path-dependent products that can only be valued by simulation. These products typically depend not only on one forward rate, but on several rates, therefore the joint conditional distribution of these rates is an essential parameter of the model. The process of estimating and parameterizing the joint conditional distribution is called *calibration*.

The inherent idea of the LIBOR market model is to model several contiguous forward rates, typically all of them of the same tenor. Following (Rebonato, 2002), they can be described by the following multi-dimensional stochastic process:

$$\frac{d\boldsymbol{f}(t)}{\boldsymbol{f}(t)} = \boldsymbol{\mu}(\boldsymbol{f}, t)dt + \boldsymbol{S}(t)d\boldsymbol{w}_{\mathbb{Q}(t)}, \tag{2.10}$$

where

- $\frac{d\boldsymbol{f}(t)}{\boldsymbol{f}(t)}$ denotes an n-column vector of percentage increments of forward rates,
- $\boldsymbol{\mu}(\boldsymbol{f}, t)$ is an n-column vector of drifts,
- $d\boldsymbol{w}_{\mathbb{Q}(t)}$ is an n-column vector of correlated Brownian motions in the measure $\mathbb{Q}$,
- $\boldsymbol{S}(t)$ is a real $n \times n$ diagonal matrix, with the i-th element, σ_i, denoting the instantaneous (percentage) volatility of the i-th forward rate.

Using Itô's formula, equation (A.1), it can be verified that a solution to the stochastic differential equation 2.10 for each forward rate f_i, $i = 1, \ldots, n$, is given by

$$f_i(t) = f_i(0) \exp\left(\int_0^t \mu_i(\boldsymbol{f}(u), u) - \frac{1}{2}\sigma_i^2(u)du + \int_0^t \sigma_i(u)dw_i(u) \right).$$

The implied Black volatility, which is the volatility that is fed into Black's formula, is the root-mean-square of the so-called *instantaneous volatility*, given by

$$\sigma^2_{\text{Black}}(T_i)T_i = \int_0^{T_i} \sigma_i^2(u)du.$$

Taking into account only one individual LIBOR rate, the model corresponds to Black's model described in section 2.2.2, where the LIBOR rate is valued under the forward-measure which makes it a martingale. For the extension to several LIBOR rates,

it is no longer possible to specify one measure that makes all rates martingales. A common measure used is the *terminal measure*, which is the forward measure of the last LIBOR rate examined. Thus, this is the only rate in the model that fulfills the assumption of lognormality and driftlessness.
Correlation among the forward rates is not yet explicitly described in the model. Instantaneous correlation, i.e. correlation among the increments of the Brownian motions is given by an $n \times n$ matrix

$$\boldsymbol{\rho} dt = d\boldsymbol{w} d\boldsymbol{w}^T,$$

or equivalently

$$\rho_{i,j} dt = dw_i dw_j.$$

Equation (2.10) can then be re-written, replacing the Brownian motion $d\boldsymbol{w}_Q(t)$ component by $\boldsymbol{B} d\boldsymbol{z}$, where B is an $n \times m$ matrix and $d\boldsymbol{z}$ are m independent Brownian motions. For each forward rate f_i this yields

$$\frac{df_i}{f_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} b_{ik} dz_k, \quad i = 1, \ldots, n, \quad m \leq n, \tag{2.11}$$

where

$$b_{ik} = \frac{\sigma_{ik}}{\sqrt{\sum_{k=1}^{m} \sigma_{ik}^2}}.$$

Some comments about this equation are appropriate:

- $\boldsymbol{B}$ is an $n \times m$ matrix, and the correlation of the Brownian motions dz_k is given by $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$. By construction, ρ is a real symmetric, positive-semidefinite matrix with ones in its diagonals, thus qualifying as a correlation matrix.

- The model is driven by $m \leq n$ independent Brownian motions, where m denotes the number of factors that drive the model. Consequently, the correlation matrix $\boldsymbol{\rho}$ has rank m.

- σ_i is the instantaneous volatility corresponding to the entries of the symmetric matrix $\boldsymbol{S}(t)$ from equation (2.10). It is related to the covariance that drive the m Brownian motions through

$$\sigma_i^2(t) = \sum_{k=1}^{m} \sigma_{ik}^2(t).$$

The model is characterized by

- the initial conditions $f_i(0)$, $i = 1, \ldots, n$,
- the choice of the instantaneous volatility function, subject to the constraint
$$\sigma^2_{\text{Black}}(T_i)T_i = \int_0^{T_i} \sigma_i^2(u)du,$$
- the choice of the number of factors m, and
- the structure of the correlation matrix $\boldsymbol{\rho}$, subject to the constraint that
$$\sum_{k=1}^{m} b_{ik}^2 = 1,$$
namely the diagonal entries of the correlation matrix $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$ must be 1.

Correlation is irrelevant when pricing products that depend only on individual LIBOR forward rates. This is, for example, the case for caplets. However, correlation is a driving factor for path-dependent instruments that span across several LIBOR rates, such as Bermudan swaptions.

Determining the instantaneous volatility and the instantaneous correlation is therefore an important aspect in the specification of the model. Solely fitting to market-observed volatilities, e.g. swaption volatilities, quickly leads to overfitting and introduces noise into the model due to illiquidity, spreads, etc. The preferred approach is to restrict the volatility function and correlation function to families of functions that are parameterized, and to choose the parameters based on market or historical data.

As an example, a parametric form of the instantaneous volatility function is given by (Rebonato, 2002):

$$\begin{aligned}
\sigma(t,T) &= g(T)f(T-t), \\
f(T-t) &= (a + b(T-t)) \cdot \exp(-c(T-t)) + d, \quad a+d > 0, \quad d, c > 0,
\end{aligned}$$

where $g(T)$ is a function that ensures that the market-given volatilities of, say, caplets are recovered.

Various functional forms for the instantaneous correlation exist. They are presented and analysed in the following chapters.

The correlation matrix in the LIBOR market model is sometimes referred to as specifying the degree of *de-correlation* among forward rates. This is due to the fact that short-rate models, where the instantaneous short rate is modelled by one Brownian motion, assume that all rates are perfectly correlated.

Chapter 3

Estimating correlation from historical swap rates

This chapter describes how, given historical market data, instantaneous correlation of the LIBOR market model can be estimated. To derive correlation from market data, Svensson's approach for modelling the yield curve is used. This allows for writing the yield curve in compact analytical form and for retrieving forward rates for arbitrary maturity and tenor within the range of the given market data. Given time series of yield curves, correlation among forward rates is then estimated.

Some remarks are in order to justify the use of historical data for estimating the correlation matrix of the LIBOR market model. The objective of the model is to evaluate complex products under the assumption of arbitrage-free markets. Therefore, evaluation under the martingale measure and thereby creating a self-financing strategy with a replication portfolio requires the use of market-traded prices and is independent of historical data. However, not all required data is available in the market. Some data may not reflect an arbitrage-free market, being influenced by distorting factors such as illiquidity or wide bid-ask spreads. This is indeed the case for instruments that are sensitive to correlation, such as swaptions, in particular for long maturities. Furthermore, there is no market-quoted instrument that is solely sensitive to instrument correlation. Therefore deriving correlation data from historical, very liquid, rates may be the preferred solution.

For evaluating the parameterizations, daily market rates from 2000 through 2004, obtained from Bloomberg, were used. Each daily sample consists of money market rates with maturities of 1, 3, 6 and 12 months, and of swap rates for maturities of 2 to 10 years, which constitutes the swap curve as given in equation (2.9). The data was split into yearly series to produce several correlation matrices that would depend on sufficiently-sized series.

3.1 Svensson's model for the yield curve

(Svensson, 1994) presents a model for estimating instantaneous forward rates, given market data. The model produces a parameterized functional form of the forward curve, which can easily be transferred into a functional form for a yield curve. Therefore, the model serves as an interpolation method given discrete market data. However, in contrast to most interpolation techniques, such as splines, the given market data is not reproduced perfectly in the model.

Svensson's model is an extension to the model presented by (Nelson and Siegel, 1987). According to Nelson and Siegel, the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Let $f(m)=f(t,t+m)$ denote the instantaneous forward rate with time to settlement m at date t. The forward rate function is expressed as

$$f(m,b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right),$$

where $b = (\beta_0, \beta_1, \beta_2, \tau_1)$, $\beta_0 > 0$, $\tau_1 > 0$, is the parameterization.

Here, the first term is constant, the second term is monotonously decreasing and the third term generates a hump-shape.

Svensson's model increases flexibility to improve the fit by extending Nelson's and Siegel's function by adding a fourth term with additional parameters β_3 and τ_2, $\tau_2 > 0$. The functional form is:

$$f(m,b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \tag{3.1}$$

where $b = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$. The parameters are categorized as follows: the first term β_0 is a constant, the second term $\beta_1 \exp\left(-\frac{m}{\tau_1}\right)$ is a monotonically decreasing function, if $\beta_1 > 0$, and monotonically increasing else. The third and fourth terms each describe a hump-shape, if $\beta_2 > 0$, or $\beta_3 > 0$, respectively, or a U-shape else. The third term has its extremum at τ_1 and the fourth term has its extremum at τ_2. The four terms and the resulting curve are depicted in figure 3.1.

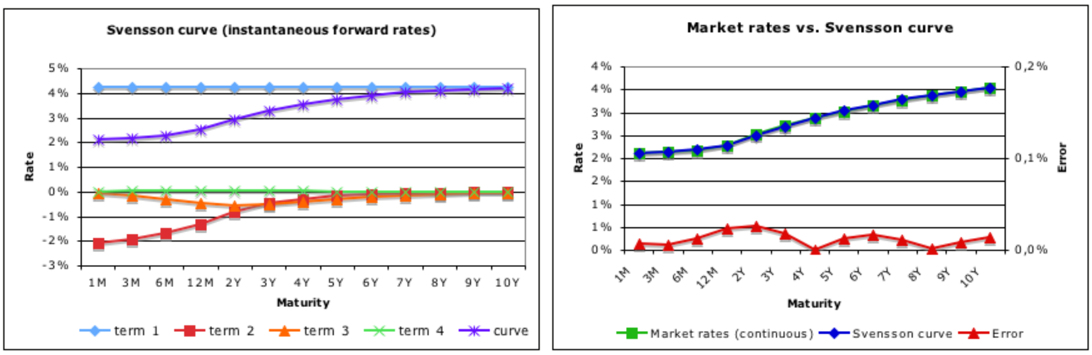


Figure 3.1: Left: The instantaneous forward rate as obtained by the Svensson model (see equation (3.1)). The four terms that add up to the curve are shown separately. Right: The spot rate as obtained by the Svensson model (see equation (3.2)). The absolute difference between market data points and interpolated points is given by the line labeled Error. The parameters are $\beta_0 = 4.245\%$, $\beta_1 = -2.1634\%$, $\beta_2 = -1.5269\%$, $\beta_3 = 0.0895\%$, $\tau_1 = 2.0236$, $\tau_2 = 1.3977$. (Source: Money market and swap rates of Feb 1, 2005, Bloomberg. Data transformed to continuous-compounding.)

According to equations (2.2) and (2.5), the continuously-compounded spot interest rate can be expressed in terms of the instantaneous forward rate:

$$R(t,T) = \frac{\int_t^T f(t,u)du}{\tau(t,T)}.$$

Let $m := \tau(t,T)$. The spot rate is then retrieved from the Svensson model by

$$\begin{aligned} R(t,T) \quad &= \quad \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left(\frac{1-\exp\left(-\frac{\tau(t,T)}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right) \right) \\ &\quad + \beta_3 \left(\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right) \right). \end{aligned} \tag{3.2}$$

An example of the estimated spot rate is shown in figure 3.1.
The discount factor curve is derived from equation (3.2) by

$$P(t,T) = e^{-mR(t,T)}. \tag{3.3}$$

Parameters are estimated by minimizing either price errors or yield errors.
For the example of figure 3.1, the market data was transformed to continuously-compounded rates before estimating the parameters:

$$r_c(t) = \begin{cases} \frac{\ln(1+r_m(t)\cdot\tau_m)}{\tau_m}, & t \leq 1 \text{ year}, \\ \frac{\ln((1+r_d(t))^{\tau_d})}{\tau_d}, & \text{else}, \end{cases}$$

where $\tau_m(t,T) = \frac{T-t}{360} \cdot \frac{365}{360}$, and $\tau_d(t,T) = T - t$ in a $\frac{30}{360}$-world. The continuously-compounded data was then fitted to the spot rate given by equation (3.2) by minimizing the mean square error:

$$E_x(\tilde{x}_1, \dots, \tilde{x}_n) = \sqrt{\frac{\sum_{i=1}^{n}(x(i) - \tilde{x}(i))^2}{n}},$$

where n is the number of market data samples, $x(i)$ is the i-th market data rate and $\tilde{x}(i)$ is the i-th rate retrieved from the Svensson model.

3.2 A parameterized functional form of the swap curve

An adapted form of Svensson's model is used to fit a parameterized form of the swap curve. The adaption is necessary as the LIBOR market model is calibrated to market-quoted data, not to continuously-compounded rates. For the correlation estimation it is especially important to be able to retrieve forward rates with various forward tenors from the model. Therefore, market day count conventions and market compounding conventions are preserved by the adapted model. For money market rates with maturities of 1, 3, 6 and 12 months the day count convention is $\frac{\text{act}}{360}$ and for swap rates of maturities $2, 3, \dots, 10$ years the day count convention is $\frac{30}{360}$.
Parameters are fitted for each data sample, which consists of a daily snapshot of a swap curve with maturities as given above. The optimal parameters for each sample are calculated by minimizing the mean square error between sample and fitted curve. As the market rates are not continuously-compounded, equation (3.2) cannot be employed for calculating the rates and thus the error. A modified equation that takes into account the market conventions is therefore used. Given discount factors $P(t,T)$, for $T \geq t$, by equation (3.3), the money market rate is given by the simply-compounded spot interest rate (see also equation (2.3)):

$$r(t,T) = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)}, \quad T \leq 1, \tag{3.4}$$

and the spot swap rate is given by (see also equation (2.7)):

$$r(t,T) = \frac{1 - P(t,T)}{\sum_{i=1}^{n} \tau(T_{i-1}, T_i) \cdot P(t, T_i)} \tag{3.5}$$

where $\tau(t,T)$ denotes the fraction in years between t and T according to the day count convention that applies. Forward rates are retrieved from the model by applying equation (2.4).

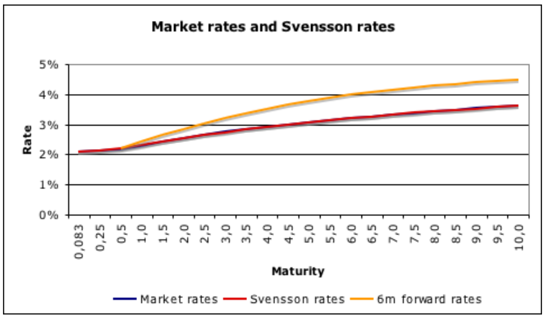


Figure 3.2: Money market and swap rates as given by market on Feb 1, 2005, and spot rates and 6-month forward rates as retrieved by Svensson model. Source: Bloomberg.

The parameterization is illustrated by an example with data of Feb 1, 2005. The market data used is shown in table 3.1. The Svensson model is parameterized with parameters shown in table 3.2. Swap rates retrieved from the model through equations (3.4) and (3.5) are given in table 3.1 together with the market rates, and forward rates with 6-month tenor retrieved from the model are given in table 3.3. Figure 3.2 shows the market rates, Svensson rates and 6-month forward rates from the examples.

3.3 Estimating instantaneous correlation of forward rates

Correlation is a linear measure of dependency between random variables. Given random variables X and Y, their correlation is given by

$$\rho_{X,Y} = \frac{E((X - E(X)(Y - E(Y))}{\sigma(X)\sigma(Y)},$$

where $E(X)$ is the expectation of X and $\sigma(X)$ is the standard deviation of X.
To estimate correlation from historical market data, forward rates are retrieved from daily swap curves as described in section 3.2. Note that the Svensson model allows for retrieving forward rates of arbitrary tenor and arbitrary maturity within the range of given market data. Daily swap curves, with maturities of 1 month to 10 years, of the years 2000 to 2004 were aggregated into yearly sets, and for each data set a series of adjacent forward rates was calculated. Here, the maturity of each forward was fixed and the time-to-maturity decreased while moving through time, as shown in figure 3.3. Denoting by τ the tenor of the forward rates, $M := 10 \times \tau$ forward rates were calculated for each daily data sample. To estimate instantaneous correlation that is consistent with the LIBOR market model, log-returns of the forward rates were

Maturity	Market rate	Svensson rate
1M	2.1051%	2.1057%
3M	2.1403%	2.1359%
6M	2.1794%	2.1832%
12M	2.2873%	2.3006%
2Y	2.5530%	2.5428%
3Y	2.7495%	2.7358%
4Y	2.9150%	2.9118%
5Y	3.0620%	3.0660%
6Y	3.1960%	3.2027%
7Y	3.3200%	3.3233%
8Y	3.4310%	3.4309%
9Y	3.5265%	3.5243%
10Y	3.6045%	3.6066%

Table 3.1: Money market and swap rates of Feb 1, 2005, from market (Source: Bloomberg) and rates as retrieved from Svensson model.

β_0	β_1	β_2	β_3	τ_1	τ_2
4.7581%	-2.6916%	-1.0994%	0.1670%	3.3479	0.0562

Table 3.2: Parameters for Svensson model for data given in table 3.1. The model is fitted to discount factors as given by equation (3.3).

Maturity	Forward rate	Maturity	Forward rate
0.5Y	2.2135%	5.5Y	3.8747%
1Y	2.4231%	6Y	3.9751%
1.5Y	2.6383%	6.5Y	4.0648%
2Y	2.8403%	7Y	4.1471%
2.5Y	3.0275%	7.5Y	4.2201%
3Y	3.2015%	8Y	4.2867%
3.5Y	3.3614%	8.5Y	4.3449%
4Y	3.5091%	9Y	4.3993%
4.5Y	3.6431%	9.5Y	4.4447%
5Y	3.7645%	10Y	4.4879%

Table 3.3: Forward rates with 6-month tenor retrieved from Svensson model parameterized to data given by table 3.1.

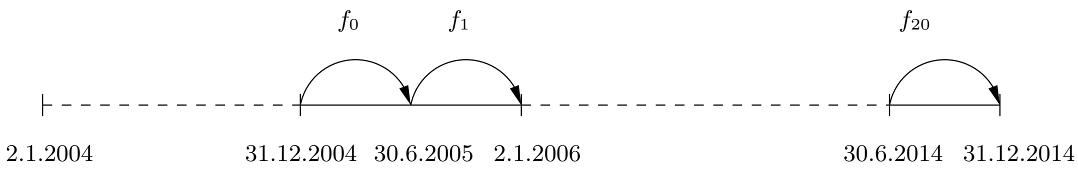


Figure 3.3: Timeline of 6-month forward rates for data set of 2000. For each trading day from Jan 2nd, 2004 to Dec 31st, 2004, forward rates with fixed maturities are retrieved, resulting in one time series per forward rate. Correlation is then calculated between log-returns of each of these series.

calculated:

$$f_i(t_j) := \ln\left(\frac{F(t_j, T_{i-1}, T_i)}{F(t_{j-1}, T_{i-1}, T_i)}\right), \quad j = 2, \dots, n,$$

where n is the number of daily forward rates. Therefore, each series of log-normal forward rate returns corresponds to a LIBOR rate of the LIBOR market model as given by equation (2.11). An $M \times M$ correlation matrix was then calculated for each yearly data set. This historical market correlation for the years 2000 to 2004 and for tenors of 3 months and 6 months is shown in figures 6.1 and 6.2, respectively.

Chapter 4

Parameterizations of the correlation matrix

A variety of parameterization functions have been introduced over the past few years that allow for expressing a given correlation matrix of forward rate correlations in a functional form. There are several advantages to this: of course, it is computationally convenient to work with an analytical formula. But also noise, such as bid-ask spreads, non-synchronous data and illiquidity are removed by focussing on general properties of correlation. Furthermore, the rank of the correlation matrix can be controlled through the functional form.

One property that is implicitly present in all parameterizations is time-homogeneity: correlation of forward rates does not depend on calendar time t, but only on the rates' time to maturity $T_i - t$.

The parameterizations presented here are divided into full-rank and reduced-rank parameterizations depending on the number of underlying Brownian motions of the model. An important aspect of these parameterizations is the number of parameters used to fit the market data. Most parameterizations advocate the use of few parameters that emphasize general properties of market correlation and prevent overfitting.

General requirements on an $M \times M$ correlation matrix $\boldsymbol{\rho}$ are:

1. $\boldsymbol{\rho}$ must be real and symmetric,
2. $\rho_{i,i} = 1$, $i = 1, \ldots, M$,
3. $\boldsymbol{\rho}$ must be positive semidefinite.

The last requirement ensures that the correlation matrix can be decomposed according to equation (2.11) into $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$ where $\boldsymbol{B}$ is an $M \times n$ matrix, $n \leq M$.
All parameterizations presented here, except for the principal component analysis approach, are parameterized by minimizing the mean square error between historical market correlation and parameterized functional form.
This chapter is organised as follows: section 4.1 explains the relationship between instantaneous correlation and terminal correlation. Full-rank parameterizations are presented in section 4.2, and a comparison of the full-rank parameterizations is given in section 4.3. Reduced-rank parameterizations are introduced in section 4.4.

4.1 Instantaneous and terminal correlation

Before moving on, the notion of correlation in the LIBOR market model should be described in a more formal way. For calibrating a LIBOR market model, instantaneous correlation is modelled. However, for pricing correlation-sensitive products, terminal correlation is used.
Define an n-dimensional LIBOR market model with m factors by equation (2.11):

$$\frac{df_i}{f_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} b_{ik} dz_k, \quad i = 1, \ldots, n, \quad m \leq n,$$

with

$$b_{ik} = \frac{\sigma_{ik}}{\sqrt{\sum_{k=1}^{m} \sigma_{ik}^2}}.$$

Definition 4.1 (Instantaneous correlation). The *instantaneous correlation* of the LIBOR rates $L(t, T_{i-1}, T_i)$ and $L(t, T_{j-1}, T_j)$ is given by the correlation of the increments of the Brownian motions:

$$\rho_{i,j}(t) = \sum_{k=1}^{m} b_{i,k} b_{j,k}, \quad i, j = 1, \ldots, n$$

Given instantaneous correlation and instantaneous volatility, terminal correlation can be computed:

Definition 4.2 (Terminal correlation). The *terminal correlation* of the LIBOR rates $L(t, T_{i-1}, T_i)$ and $L(t, T_{j-1}, T_j)$ is given by

$$\tilde{\rho}_{i,j}(t) = \frac{\int_0^t \sigma_i(u)\sigma_j(u)\rho_{i,j}(u)du}{\sqrt{\int_0^t \sigma_i(u)^2 du \int_0^t \sigma_j(u)^2 du}}. \tag{4.1}$$

If instantaneous volatilities are not constant, they have a significant impact on terminal correlation and can produce terminal de-correlation, even in the case of perfect instantaneous correlation.
Prices of correlation-sensitive products depend on terminal correlation, and thus instantaneous correlation and instantaneous volatility. It is important to note that there is no instrument that is sensitive solely to instantaneous correlation. Therefore, estimating correlation from a product that is sensitive to multiple factors is not straight-forward and can lead to ambiguous results. As an example, (Rebonato, 2002, p.175) considers swaption prices that terminal correlation can be inferred from.
In the case of constant volatilities, terminal correlation is just the average correlation over the period. To see this, denote by σ_i and by σ_j the constant volatilities $\sigma_i(u)$ and $\sigma_j(u)$. Then, the volatilities can be factored out from the integrals:

$$\begin{aligned}
\bar{\rho}_{i,j}(t) &= \frac{\int_0^t \sigma_i(u)\sigma_j(u)\rho_{i,j}(u)du}{\sqrt{\int_0^t \sigma_i(u)^2du \int_0^t \sigma_j(u)^2du}} \\
&= \frac{\sigma_i\sigma_j \int_0^t \rho_{i,j}(u)du}{\sqrt{t\sigma_i t\sigma_j}} \\
&= \frac{\int_0^t \rho_{i,j}(u)du}{t}.
\end{aligned}$$

See also (Rebonato, 2004, pp.141) for an in-depth discussion of the relationship between instantaneous and terminal correlation.

4.2 Full-rank parameterizations of the correlation matrix

Pairwise correlation of M random variables is described by an $M \times M$ matrix. Since a correlation matrix $\boldsymbol{\rho}$ is required to be symmetric, positive-semidefinite and to have ones in its diagonals, it is characterized by $\frac{M(M-1)}{2}$ entries. For practical purposes, it is often desirable to describe the correlation matrix with fewer entries or to describe its entries through a parameterized functional form. Furthermore, a correlation matrix estimated from market data may not fulfil the property of being positive definite, as it is not guaranteed to be of full rank. However, the rank of the matrix reflects the number of independent Brownian motions that drive the LIBOR market model, and therefore the rank of the matrix should be controllable. Fitting estimated correlation data to an appropriate functional form guarantees that the requirement of

positive semidefiniteness of the correlation matrix is fulfilled. In the case of full-rank parameterizations, the resulting matrix is positive definite.
Furthermore, a functional form allows for emphasizing general properties of the correlation data. This may be favoured over an exact fit to a data set that may not be a particularly good representative.
Some full-rank parameterizations are presented and examined, and results obtained from fitting market data to these functional forms are given in section 6.3 (see tables 6.1 to 6.5, figures 6.3 to 6.12 for 3m-forward rate correlations and 6m-forward rate correlations).
Each full-rank parameterization described here was fitted by minimizing the mean square error:

$$E_{\boldsymbol{\rho}}(\tilde{\boldsymbol{\rho}}) = \sqrt{\frac{\sum_{i=1}^{M}\sum_{j=1}^{M}|\rho_{ij} - \tilde{\rho}_{ij}|^2}{M^2}}. \tag{4.2}$$

A modified version of the mean square error function assigns weights to each individual error term, in order to emphasize the error of values that are considered to be significant for the correlation structure. However, such a penalty function was not considered for the results obtained here. See (Rebonato, 2002, pp. 259) for a discussion of choosing a penalty function with regard to the instrument to be priced on the model.
The most straight-forward functional form is given, amongst others, by (Rebonato, 2004):

Definition 4.3 (One-parameter parameterization). A simple functional form for a correlation function is defined by:

$$\rho_{i,j} = \exp(-\beta|i-j|), \quad \beta > 0. \tag{4.3}$$

One should note that this equation contains some common notational abuse; the correct formal expression is

$$\rho_{i,j} := \rho_{i,j}(t) = \exp(-\beta|T_i - t, T_j - t|), \quad \beta > 0.$$

For ease of notation, the parameterizations will be defined using the notation of equation (4.3). It should be kept in mind that the parameters i and j actually reference the time to maturity of the i-th and j-th forward rates.
Equation (4.3) always produces a valid correlation matrix in the sense that it produces a real, symmetric, positive-definite matrix. However, correlation is only dependent

on the distance between maturities and is constant with regard to t. Under the assumption of constant volatilities, instantaneous and terminal correlation are equal. β is called the *de-correlation factor* or *rate of de-correlation* as it controls the decrease in correlation with increasing maturity interval. Setting $\beta := 0$ results in a model with perfect instantaneous correlation, thereby reducing the number of driving factors of the model to 1.

An overview of more sophisticated approaches, classified into full-rank and reduced-rank parameterizations, is given by (Brigo, 2002). These and others are presented below.

A family of full-rank parameterizations, called *semi-parametric* parameterizations, is proposed by (Schoenmakers and Coffey, 2000) by defining correlation through a finite sequence of positive real numbers $1 = c_1 < c_2 < \ldots < c_M$ and setting

$$\rho_{i,j} := \frac{c_i}{c_j}, \quad i \leq j, \quad i,j = 1,\ldots,M.$$

Such a matrix $\boldsymbol{\rho}$ fulfills two criteria:

Definition 4.4 (Conditions on correlation matrix). The following conditions can be observed on correlation estimated from market data :

(i) Correlation decreases for increasing maturity intervals.

(ii) The forward curve tends to flatten and correlation increases with large maturities. Formally, for a constant p it holds that $\rho_{i,i+p}$ increases with increasing i.

Note that the one-parameter functional form given by equation (4.3) fulfills condition (i), but fails on condition (ii).

The c_i can be characterized by a sequence of non-negative numbers $\Delta_2, \ldots, \Delta_M$:

$$c_i = \exp\left(\sum_{j=2}^{i} j\Delta_j + \sum_{j=i+1}^{M} (i-1)\Delta_j\right). \tag{4.4}$$

Parameterizations are then obtained by imposing restrictions on the Δ's.

Definition 4.5 (Two-parameters parameterization). Setting $\Delta := \Delta_2 = \cdots = \Delta_{M-1}$, given Δ_M and setting $\rho_\infty := \rho_{1,M}$ and $\eta := \Delta\frac{(M-1)(M-2)}{2}$, a stable parameterization with two parameters that yields a full-rank correlation matrix $\boldsymbol{\rho}$ is given by

(Schoenmakers and Coffey, 2000)[1]:

$$\rho_{i,j} = \exp\left(-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{M-i-j+1}{M-2}\right)\right), \tag{4.5}$$

$0 < \rho_\infty$, $0 \le \eta \le -\ln\rho_\infty$, $i, j = 1, \ldots, M$.

This formulation is stable in the sense that relatively small movements in the c-parameters cause only relatively small changes in ρ_∞ and η.
ρ_∞ denotes the correlation between the forward rates with highest distance, which is also the minimal correlation possible in the model. η is the rate of de-correlation. Due to the dependence on the parameters i and j, the factor of η fulfills condition (ii) of definition 4.4. Condition (i) is fulfilled by the dependence on the factor $|i-j|$. In the following, this parameterization will not be examined further, as (Schoenmakers and Coffey, 2000) present a better fitting parameterization:

Definition 4.6 (Improved two-parameters parameterization). Assuming that the Δ_i follow a straight line for $i = 1, \ldots, M-2$ and setting $\Delta_{M-1} = 0$ results after a change of variable in a stable, full-rank, two-parameters parameterization with

$$\rho_{i,j} = \exp\left(-\frac{|j-i|}{M-1}\left(-\ln\rho_\infty + \eta\cdot f(i,j,M)\right)\right), \tag{4.6}$$

where

$$f(i,j,M) = \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)},$$

and $0 < \rho_\infty$, $0 \le \eta \le -\ln\rho_\infty$, $i, j = 1, \ldots, M$.

Here, η is related to the steepness of the straight line through the Δ_i's. Setting $\eta := 0$, the parameterization collapses to an identical structure as the one-parameter parameterization.
Keeping the maturity interval $|j-i|$ constant, $f(i,j,M)$ is monotonically decreasing with growing maturity, and since η is a positive constant, $f(i,j,M)$ is the decay rate of η. Still keeping the maturity interval constant, $\rho_{i,j}$ is therefore increasing with increasing maturity. This observation makes the parameterization fulfil condition (ii) of definition 4.4. Figure 4.1 shows an example of $f(i,i+1,M)$ for increasing i.

[1]The formula given in (Brigo, 2002) reads $\rho_{i,j} = \exp\left(-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{M-1-i-j}{M-2}\right)\right)$. Appendix B shows that the formula given by (Schoenmakers and Coffey, 2000) is indeed the correct formula.

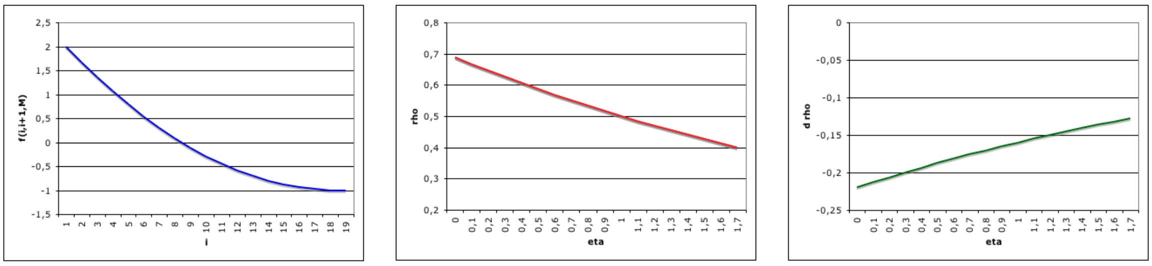


Figure 4.1: Example of change in parameter η of improved two-parameters parameterization of equation (4.5). Left: How the factor $f(i, j, M)$ evolves for constant maturity interval $|j - i| = 1$. Middle: Let $\rho_\infty = 0.1697$, $i = 1$, $j = 5$. $\rho_{i,j}$ is plotted for $0 \leq \eta \leq -\ln \rho_\infty$. Right: The figure shows the partial derivative of ρ with respect to η (see equation (4.7) with ρ_∞, i and j defined as in the previous example).

The effect of a change in η is best described by the partial derivative of the equation with respect to η:

$$\frac{\partial \rho_{i,j}}{\partial \eta} = -\frac{|i-j|}{M-1} f(i,j,M) \exp\left(\frac{-|i-j|}{M-1}(-\ln \rho_\infty + \eta \cdot f(i,j,M))\right), \tag{4.7}$$

where $0 \leq \eta \leq -\ln \rho_\infty$. Increasing η increases the de-correlation of $\rho_{i,j}$. Figure 4.1 contains examples that show the effect on $\rho_{i,j}$ of changing η and the partial derivative with respect to η.

A straight-forward extension of the one-parameter parameterization of equation (4.3) is suggested by (Rebonato, 2004):

Definition 4.7 (Classical two-parameters parameterization). A two-parameters parameterization is given by

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp(-\beta |i-j|), \quad \beta \geq 0. \tag{4.8}$$

De-correlation of forward rates with increasing distance does not tend towards zero, but towards ρ_∞ instead. ρ_∞ represents asymptotically the correlation between the rates with highest distance. Apart from this, all observations of the one-parameter parameterization of definition 4.3 apply.

To fulfil condition (ii), (Rebonato, 1999b) suggests modifying the constant β to become a function $\beta_{\max i,j}$ that decays with increasing distance. This can be expressed as a function of two constants:

$$\beta_{\max i,j} := \beta - \alpha \max(i,j),$$

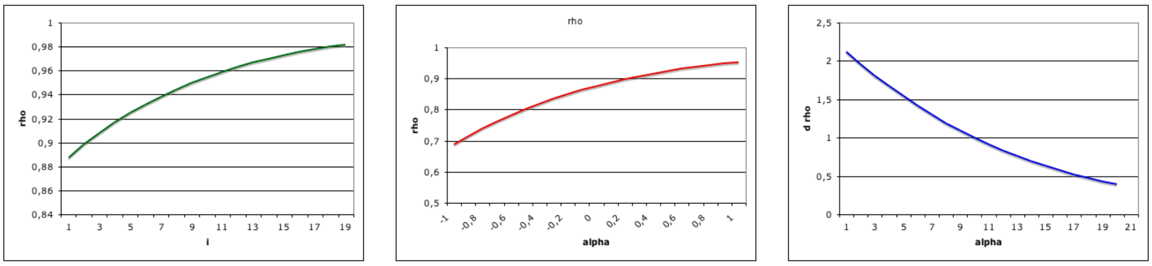


Figure 4.2: Example of change in parameter α of three-parameters parameterization of equation (4.10). Left: How $\rho_{i,j}$ changes for constant maturity interval $|i-j| = 1$; parameters are: $\beta = 0.092$ and $\alpha = 0.1034$. Middle: How $\rho_{i,j}$ evolves for increasing α; $\beta = 0.092$. Right: The partial derivative of $\rho_{i,j}$ with respect to α (see equation (4.11), with $\beta = 0.092$, $i = 1$, $j = 2$).

and yields

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp(-|i-j|(\beta - \alpha(\max(i,j)))), \tag{4.9}$$

where $-1 \le \rho \le 1$, $\beta > 0$, $0 \le \alpha \le \frac{\beta}{M}$.
(Schoenmakers and Coffey, 2000) point out that this functional form may fail to define a valid correlation structure, as not all eigenvectors of the matrix $\boldsymbol{\rho}$ are guaranteed to be positive, and therefore, the matrix is not guaranteed to be positive definite.
(Rebonato, 2004) provides a further functional form for β that fixes this problem:

Definition 4.8 (Three-parameters parameterization). Substituting the constant β of equation (4.8) with the function $\beta_{\min(i,j)} := \beta\exp(-\alpha\min(i,j))$, yields a symmetric, positive definite correlation function

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp(-|i-j|\beta\exp(-\alpha\min(i,j))), \tag{4.10}$$

where $-1 \le \rho_\infty \le 1$, $\beta > 0$, $\alpha \in \mathbb{R}$.

Again the minimum correlation tends asymptotically towards ρ_∞, and $\beta\exp(-\alpha\min(i,j))$ controls the rate of decay, where the expression itself decreases at a rate of $\exp(-\alpha\min(i,j))$. Due to this second decay constant, $\rho_{i,j}$ increases with growing maturity and constant maturity interval. The special case of $\alpha = 0$ reduces this parameterization to the classical two-parameters parameterization of definition 4.7.
Simplifying by setting $\rho_\infty = 0$, the partial derivative of $\rho_{i,j}$ with respect to α is given by:

$$\frac{\partial\rho_{i,j}}{\partial\alpha} = (\min(i,j)\cdot|i-j|)\exp(-\alpha\min(i,j) - |i-j|\beta\exp(-\alpha\min(i,j))). \tag{4.11}$$

The derivative is always positive, therefore, $\rho_{i,j}$ increases with regard to increasing α. Figure 4.2 contains examples that show how $\rho_{i,j}$ increases with increasing maturity and constant maturity interval $|j-i|$, how $\rho_{i,j}$ evolves with growing α and how the derivative of $\rho_{i,j}$ with regard to α evolves.
(Rebonato, 2004) suggests a further functional form that he calls the square-root model:

Definition 4.9 (Two-parameters, square-root parameterization). The functional form for the square-root parameterization is

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp(-\beta|\sqrt{i}-\sqrt{j}|), \tag{4.12}$$

where $-1 \leq \rho_\infty \leq 1$, $\beta > 0$.

Both conditions of definition 4.4 are fulfilled.
The factor $|\sqrt{i}-\sqrt{j}|$ is monotonically decreasing for constant interval $|i-j|$. In turn, $\rho_{i,j}$ increases for constant interval and increasing maturity. As in previous parameterizations, β is the rate of de-correlation.

4.3 Comparison of full-rank parameterization methods

All parameterization functions presented above aim at fitting given correlation data in such a way that general observations on market data are fulfilled. At the same time, overfitting is to be avoided as only generalized aspects of correlation data are to be isolated. Therefore, the parameterizations given are very general in nature, and it is not expected that they accurately represent a specific given data set. Mean square errors of the parameterizations for the market data lie between 5.78% and 22.70%.
There is general agreement (see (Schoenmakers and Coffey, 2000), (Brigo, 2002), (Rebonato, 2004)) that the two conditions stated in definition 4.4 are general features of a market correlation structure. All methods satisfy condition (i). Fulfilling condition (ii) may come at a computational cost, e.g. when calculating terminal correlation as given in equation (4.1). Correlation that depends solely on the time interval of i and j can be factored out of the integral that defines terminal correlation, making the integral itself more tractable. Thus, parameterizations defined by equations (4.3) and (4.8) may be simple, but convenient to work with.
Depending on the data given, it may occur that the factor that drives condition (ii) becomes zero or close to zero. This is typically the case when correlation is not only

high at long maturities but also at short maturities. An example for ill-fitting data with regard to condition (ii) is the market data of 2004.
Another distinguishing aspect of the parameterization methods is the fact whether they allow for negative correlation or not. In general, the approaches by (Schoenmakers and Coffey, 2000) do not allow for negative correlation, whereas the approaches by (Rebonato, 2004) permit negative correlation. This property is driven by the variable ρ_∞, which denotes asymptotically the minimum correlation that the model allows for.

4.4 Reduced-rank parameterizations of the correlation matrix

A different approach to fitting the correlation matrix is to parameterize to a matrix that has not full rank. In this case, the model is driven by fewer independent Brownian motions than LIBOR rates. More formally, the correlations in the LIBOR market model are given by

$$\rho_{i,j}(t) = dW_i(t)dW_j(t), \quad i, j = 1, \dots, M,$$

where the correlation matrix $\boldsymbol{\rho}$ can be decomposed into matrices

$$\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T.$$

This is achieved by $dW(t) = B(t)dU(t)$, where $U_1, \dots, U_n$ are independent Brownian motions and B is an $M \times n$ matrix. It then holds that $dW_i(t) = \sum_{j=1}^{m} b_{i,j}(t)dU_j(t)$ and $\rho_{i,j}(t) = \sum_{k=1}^{m} b_{i,k}(t)b_{j,k}(t)$.
The eigen decomposition, also called spectral decomposition, of a square $M \times M$ matrix $\boldsymbol{A}$ is given by $\boldsymbol{\rho} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where $\boldsymbol{P} = (\boldsymbol{X}_1\boldsymbol{X}_2 \cdots \boldsymbol{X}_M)$ is the matrix composed of eigenvectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_M$, and $\boldsymbol{D}$ is a diagonal matrix containing the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_M$ as diagonal entries. Let $\boldsymbol{\Lambda}$ be the diagonal matrix whose entries are the square roots of the entries in $\boldsymbol{D}$, and let $\boldsymbol{A} := \boldsymbol{P}\boldsymbol{\Lambda}$. Then $\boldsymbol{\rho} = \boldsymbol{A}\boldsymbol{A}^T$. In the case of $\boldsymbol{A}$ being a non-singular real matrix, $\boldsymbol{\rho}$ is symmetric. Instead of defining $\boldsymbol{\rho}$ through an $M \times M$ matrix $\boldsymbol{A}$, it can be defined by a real n-rank $M \times n$ matrix $\boldsymbol{B}$, such that $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$.
One method of reducing the rank of the correlation matrix is to apply principal component analysis. Another method is the hypersphere decomposition, where the reduced-rank matrix is formulated in terms of angles. Results from given market data are given in section 6.4.

In contrast to the earlier approaches, reducing the rank through principal component analysis does not yield a compact parametric formulation for the correlation matrix entries. The number of parameters is specified by the $M \times n$ matrix of the most significant eigenvectors and n eigenvalues. The approach is described by (Rebonato, 1999b) and by (Rebonato, 2002) as a mechanism to create a positive-semidefinite correlation matrix from market data that may not fulfil the property of positive-semidefiniteness. It is summarized by (Brigo, 2002), and also used by (Alexander and Lvov, 2003) and (Alexander, 2003).

Definition 4.10 (Rank-reduction through principal component analysis). Let $\mathbf{\Lambda}^{(n)}$ be the matrix $\mathbf{\Lambda}$ with the $M - n$ smallest diagonal entries set to 0. Then, $\boldsymbol{B}^{(n)} := \boldsymbol{P}\mathbf{\Lambda}^{(n)}$ and $\tilde{\boldsymbol{\rho}}^{(n)} = \boldsymbol{B}^{(n)}\boldsymbol{B}^{(n)^T}$. For $\tilde{\boldsymbol{\rho}}^{(n)}$ to yield a valid correlation matrix, the entries are rescaled to ensure that the diagonal entries are 1:

$$\rho_{ij}^{(n)} := \frac{\tilde{\rho}_{ij}^{(n)}}{\sqrt{\tilde{\rho}_{ii}^{(n)}\tilde{\rho}_{jj}^{(n)}}}.$$

The eigenvector of $\boldsymbol{\rho}$ with highest corresponding eigenvalue is called the *principal component* and among all eigenvectors it has most significant influence on the correlation matrix $\boldsymbol{\rho}$. Taking into consideration the n eigenvectors corresponding to the highest n eigenvalues results in an n-rank correlation matrix with minimum information disposed.

(Alexander and Lvov, 2003) use a principal component analysis with three eigenvectors and report that, in the case of the correlation matrix, the eigenvectors can be classified into trend, tilt and curvature.

Another approach of a reduced-rank parameterization is given by (Rebonato, 1999b) and (Rebonato, 2002). Here, the entries of $\boldsymbol{B}$ are described as angles.

Definition 4.11 (Rank-reduction through hypersphere decomposition). An n-rank matrix is defined by $\boldsymbol{\rho}^{(n)}(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\theta})^T$ with

$$b_{ij}(\boldsymbol{\theta}) = \begin{cases} \cos_{ij} \prod_{k=1}^{j-1} \sin\theta_{ik}, & j = 1, \ldots, n-1 \\ \prod_{k=1}^{j-1} \sin\theta_{ik}, & j = n, \end{cases}$$

$i = 1, \ldots, M$. $\boldsymbol{\theta}$ is defined through $M(n-1)$ angular coordinates θ_{ij}. A special form is to reduce the number of parameters by choosing $\theta_{ij} := \theta_i$, $\forall j = 1, \ldots, n-1$.

The matrix $\boldsymbol{\rho}^{(n)}$ is guaranteed to be symmetric, positive-semidefinite and with ones in the diagonals. The row vectors of $\boldsymbol{B}$ can be viewed as coordinates lying on a unit

hypersphere[2]. The angles θ_{ij} are chosen by minimizing the mean square error as given by equation (4.2).
It is important to note that the number of parameters $\boldsymbol{\theta}$ in the general case is $M(n-1)$ and it is M in the special case where there is only one parameter per row, i.e. $\theta_{ij} := \theta_i,\ j = 1, \dots, n-1$. By fitting to a full-rank matrix, the number of parameters would increase from $M\frac{(M-1)}{2}$ significant entries in the original matrix to $M(M-1)$. Therefore, to achieve a decrease of the number of parameters, it must hold that $n << \frac{M-1}{2}$.
(Brigo, 2002) gives a compact expression of $\boldsymbol{\rho}^{(n)}$ for the special case $\theta_{ij} := \theta_i,\ j = 1, \dots, n-1$:

$$\rho_{ij}^{(n)} = \sqrt{(1-\alpha_i^2)(1-\alpha_j^2)}\frac{1-(\alpha_i\alpha_j)^{n-1}}{1-\alpha_i\alpha_j} + (\alpha_i\alpha_j)^{n-1}, \quad \alpha_k := \sin\theta_k \in [-1,1]. \quad (4.13)$$

This special case was applied when fitting to historical market data (see figures 6.13 to 6.21).
By construction, setting $\alpha_i := \alpha,\ \forall i$, yields a correlation matrix with all entries 1, which is identical to a one-factor LIBOR market model. In this case, terminal correlation depends solely on instantaneous volatility (see equation 4.1).

[2] An n-hypersphere with radius R is defined as an n-tuple $(x_1, \dots, x_n)$ such that $x_1^2 + \dots + x_n^2 = R^2$. Since this condition is satisfied for the row vectors of $\boldsymbol{B}$, the diagonal entries of $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$ are 1.

Chapter 5

Relationships between correlation of different LIBOR tenors

This chapter analyses the relationship of correlation matrices that were applied to forward rates of different tenors. Under certain circumstances it may be desirable to apply an existing parameterization to rates of different tenor. Consider as an example LIBOR rates that typically have a tenor of 6 months and swaptions that have a tenor of 1 year. A generalized correlation structure or parameterization makes the need for calibrating to various tenors obsolete. It is important to note that all generalizations are approximations only. This is mainly due to the fact that rates of different tenors cannot be assumed to follow a driftless Geometric Brownian motion under the same measure at the same time (this point was already made in section 2.2.3). Application to market data shows that, depending on the parameterization, errors are tolerable.
(Brigo and Mercurio, 2001) and (Alexander, 2003) present an approach for estimating the volatility of two annual forward rates given their corresponding semi-annual forward rates. This approach is most useful when using historical market correlation or when using a parameterization that reproduces historical market correlation well, such as principal component analysis or hypersphere decomposition. The approach is explained in section 5.1.
For deducing semi-annual forward rate correlations from annual forward rate correlations, the correlation of semi-annual rates that span one given annual rate must be estimated. Good results were achieved by making the full-rank parameterizations from section 4.2 tenor-independent. This works well for parameterizations with few parameters as they do not reproduce the actual correlation data, but fit the data to

generalized properties that are present in the correlation matrix. The approach does not lend itself for use on historical market correlation or near-exact fits. It is presented in section 5.2.

5.1 Determining instantaneous correlation for rates of larger tenors

(Brigo and Mercurio, 2001) present an approximation for the volatility for forward rates of annual tenor, given the volatilities of two forward rates with semi-annual tenor. (Alexander, 2003) extends this approach to determining the correlation of two annual forward rates given the corresponding semi-annual forward rates. A straight-forward formula for correlation can be derived using the one-parameter parameterization from definition 4.3. The derivation is shown here, followed by an application to the historical market data from chapter 4. Without loss of generality, the derivation can be applied to other tenor changes, e.g. quarterly to semi-annual forward rates.
Denote by $F(t) := F(t; T_0, T_2)$ the forward rate with annual tenor between two points in time T_0 and T_2, and by $f_1(t) := F(t; T_0, T_1)$ and $f_2(t) := F(t; T_1, T_2)$ the corresponding forward rates with semi-annual tenor. Assume that tenors are constant, i.e. $\tau(T_0, T_2) = 1$ and $\tau(T_0, T_1) = \tau(T_1, T_2) = \frac{1}{2}$.
The annual forward rate can then be expressed in terms of the semi-annual forward rates using equation (2.4)

$$
\begin{aligned}
F(t) &= \frac{P(t,T_0)}{P(t,T_2)} - 1 \quad = \quad \frac{P(t,T_0)}{P(t,T_1)} \frac{P(t,T_1)}{P(t,T_2)} - 1 \\
&= \left(\frac{f_1(t)}{2} + 1 \right) \left(\frac{f_2(t)}{2} + 1 \right) - 1 \\
&= \frac{f_1(t) f_2(t)}{4} + \frac{f_1(t) + f_2(t)}{2}.
\end{aligned}
$$

Assuming lognormal forward rates for $f_1(t)$ and $f_2(t)$, it follows that $F(t)$ cannot exactly be lognormal at the same time.
The semi-annual forward rates are modelled by SDEs

$$
\begin{aligned}
df_1(t) &= (...)dt + \sigma_1(t) f_1(t) dZ_1(t) \\
df_2(t) &= (...)dt + \sigma_2(t) f_2(t) dZ_2(t),
\end{aligned}
$$

where the two Brownian motions are correlated with $\rho(t)$. The annual forward rate can be expressed by the two semi-annual forward rates:

$$\begin{aligned}
dF(t) &= \frac{df_1(t)f_2(t)}{4} + \frac{df_1(t)}{2} + \frac{df_2(t)}{2} \\
&= \frac{f_2(t)}{4}((...)dt + \sigma_1(t)f_1(t)dZ_1) + \frac{f_1(t)}{4}((...)dt + \sigma_2(t)f_2(t)dZ_2) \\
&+ \quad (...)dt + \sigma_1(t)\frac{f_1(t)}{2}dZ_1 + (...)dt + \sigma_2(t)\frac{f_2(t)}{2}dZ_2 \\
&= (...)dt + \sigma_1(t)\left(\frac{f_1(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)dZ_1(t) + \sigma_2(t)\left(\frac{f_2(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)dZ_2(t).
\end{aligned}$$

Denote by $\sigma(t)$ the percentage volatility of $F(t)$, conditional on the information available at t. The variance of $F(t)$ is then given by

$$\begin{aligned}
\sigma^2(t)F(t)^2 &= \sigma_1^2(t)\left(\frac{f_1(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)^2 + \sigma_2^2(t)\left(\frac{f_2(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)^2 \\
&+ \quad 2\rho(t)\sigma_1(t)\sigma_2(t)\left(\frac{f_1(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)\left(\frac{f_2(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right).
\end{aligned}$$

Setting

$$\begin{aligned}
u_1(t) &:= \frac{1}{F(t)}\left(\frac{f_1(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right) \\
u_2(t) &:= \frac{1}{F(t)}\left(\frac{f_2(t)}{2} + \frac{f_1(t)f_2(t)}{4}\right)
\end{aligned}$$

yields

$$\sigma^2(t) = u_1^2(t)\sigma_1^2(t) + u_2^2(t)\sigma_2^2(t) + 2\rho(t)\sigma_1(t)\sigma_2(t)u_1(t)u_2(t). \tag{5.1}$$

It is now assumed that $u_1(t) = u_1(0) = u_2(t) = u_2(0)$ for all t, and that the two semi-annual forward rates have the same volatility (the volatility may change for different pairs of semi-annual forward rates). Equation (5.1) can then be re-written as

$$\sigma^2(t) \approx 2u^2\sigma_1^2(t)(1 + \rho(t)).$$

To calculate the covariance, note that $F_i(t)$ refers to the i-th annual forward rate in a series of annual forward rates from t, and that $f_i(t)$ refers to the i-th semi-annual forward rate in a series of semi-annual forward rates. Therefore, the annual forward rate $F_i(t)$ corresponds to the semi-annual forward rates $f_{2i-1}(t)$ and $f_{2i}(t)$.
The covariance of two sums is the sum of its covariances, therefore the covariance between two annual forward rates $F_i(t)$ and $F_j(t)$ is given by

$$\hat{\sigma}_{i,j}(t) \approx u^2\sigma_{2i}(t)\sigma_{2j}(t)(\rho_{2i-1,2j-1}(t) + \rho_{2i-1,2j}(t) + \rho_{2i,2j-1}(t) + \rho_{2i,2j}(t)),$$

and the correlation $\hat{\rho}_{i,j}(t)$ is then given by

$$\hat{\rho}_{i,j}(t) \approx \frac{\rho_{2i-1,2j-1}(t) + \rho_{2i-1,2j}(t) + \rho_{2i,2j-1}(t) + \rho_{2i,2j}(t)}{2\sqrt{(1+\rho_{2i-1,2i}(t))(1+\rho_{2j-1,2j}(t))}}. \tag{5.2}$$

If one assumes that correlation depends only on the time interval between T_i and T_j, denoted by $\rho_{i,j}(t) := \rho_{|i-j|}$, this reduces to

$$\hat{\rho}_{i,j} \approx \frac{2\rho_{2|i-j|} + \rho_{2|i-j-1|} + \rho_{2|i-j+1|}}{2(1+\rho_1)}.$$

In the case of the one-parameter parameterization from definition 4.3, which is given by:

$$\rho^{|i-j|} := \exp(-\beta|i-j|), \quad \beta > 0,$$

this yields

$$\begin{aligned}
\hat{\rho}_{i,j}(t) &\approx \frac{2\rho^{2|i-j|} + \rho^{2|i-j|-1} + \rho^{2|i-j|+1}}{2(1+\rho)} \\
&= \frac{\rho^{2|i-j|-1}(2\rho + 1 + \rho^2)}{2(1+\rho)} \\
&= \frac{\rho^{2|i-j|-1}(1+\rho)}{2}.
\end{aligned}$$

This result can be explained by the fact that, given that correlation is equal for constant time intervals $|i-j|$, correlation for annual forward rates is the average of correlation of semi-annual rates of time intervals $2|i-j|$ and $2|i-j|-1$.

Closed formulas for the other parameterizations of chapter 4 can be calculated using equation (5.2).

Given the full-rank parameterized correlation matrices of 3-month forward rates (see tables 6.1 to 6.5), correlation of 6-month forward rates was calculated using equation (5.2). The mean square errors of the resulting correlation matrices and the original parameterized correlation matrices of 6-month forward rate tenor are given in table 6.11.

In the case of historical market correlation, mean square errors are well below 0.5%. For the parameterizations, errors are between 0.75% and 1.5%.

5.2 Tenor-independence

For determining correlations of semi-annual forward rates given correlations of annual rates (or any other tenor pair $\frac{1}{2}\tau(T_1,T_2)$ vs. $\tau(T_1,T_2)$), the route outlined in the previous section cannot be taken. From the point of view of forward rates of semi-annual

tenor, the correlation of forward rates of annual tenor entails a loss of information, namely the correlation between adjacent forward rates that span an annual rate.
The approach proposed here is to make the full-rank parameterizations introduced in section 4.2 tenor-independent. There are two ways to achieve this: Correlations of different tenors are retrieved from a given parameterization by a transformation function, or parameters are fitted to a normalized version of the parameterization function. Both methods yield the same results.
The first method may be less tractable for more complex parameterization functions, but it is a useful approach to analysing the impact of obtaining correlations of a tenor other than the data used for fitting. The second method then specifies in a straight-forward way how the correlations are obtained.
Essentially, tenor-independent correlation parameterization is an interpolation of the correlation surface, where the original mapping $\mathbb{Z} \times \mathbb{Z} \mapsto [-1, 1]$ is transformed to a mapping $\mathbb{Q} \times \mathbb{Q} \mapsto [-1, 1]$, converging towards the limiting case of infinitely many forward rates with infinitesimal tenor.
Let $\hat{\rho}_{i,j}$, $i, j = 1, \ldots, M$, denote a correlation matrix of forward rates of semi-annual tenor, and $\rho_{i,j}$, $i, j = 1, \ldots, 2M$ correlations of forward rates of quarterly tenor. A tenor-independent parameterization then implies that

$$\hat{\rho}_{i,j} = \rho_{2i,2j}, \quad i, j = 1, \ldots, M. \tag{5.3}$$

This is a special case of equation (5.2), where it is assumed that the correlation of adjacent forward rates $\rho_{2i-1,2i}$ and $\rho_{2j-1,2j}$ is 1, and that $\rho_{2i-1,2j-1} = \rho_{2i-1,2j} = \rho_{2i,2j-1} = \rho_{2i,2j}$. This is the main assumption of the tenor-independent approach, namely that correlation between two forward rates depends only on the maturity of the forward rates, and not on their tenor (hence the name). This is an approximation, and its implications should be carefully examined: in the previous section it was shown how to derive the correlation of forward rates of annual tenor from semi-annual forward rate correlations, with the annual correlation $\hat{\rho}_{i,j}$ depending on all semi-annual correlations spanning the two annual forward rates, namely $\rho_{2i-1,2j-1}$, $\rho_{2i-1,2j}$, $\rho_{2i,2j-1}$, $\rho_{2i,2j}$, $\rho_{2i-1,2i}$ and $\rho_{2j-1,2j}$. Using the approach proposed here, the contribution to the correlation of the two annual forward rates depends solely on $\rho_{2i,2j}$.
Consequently, applying this approximation directly to market data leads to relatively large errors. In table 6.12, the row “Historical market correlation” shows the mean square error of calculating semi-annual forward rate correlations from quarterly forward rate correlations using equation (5.3), where the maximum error is 3.14%. Sim-

ilar results are obtained by applying equation (5.3) to data obtained from a parameterization that involves a large number of parameters, such as principal component analysis and hypersphere decomposition. This can be attributed to the fact that these parameterizations reproduce the market data very well, and that this correlation structure is dependent on the tenor of all rates involved.

It is now examined how this approach can be applied to the full-rank parameterizations of section 4.2.

Generally, to obtain the correlation of $\rho_{2i+1,2j}$, namely the correlation of two adjacent quarterly rates that span one semi-annual rate, any interpolation technique can be employed, as long as it fulfills equation (5.3). The simplest such technique is linear interpolation between $\hat{\rho}_{i,j}$ and $\hat{\rho}_{i+1,j}$.

However, since the values $\hat{\rho}_{i,j}$, $i, j = 1, \ldots, M$, are specified by a parameterized function, this function is a natural candidate for such an interpolation function. As all full-rank parameterization functions given in chapter 4 are continuous and monotonous functions, their properties that make them candidates for correlation functions in the first place are preserved across tenor changes.

It is therefore proposed to employ the parameterization function that the original correlation matrix is composed of for interpolation.

As an example consider the one-parameter parameterization of definition 4.3, where

$$\hat{\rho}_{i,j} := \hat{\rho}(i, j, \hat{\beta}) = \exp(-\hat{\beta}|i - j|), \quad \hat{\beta} > 0.$$

Setting $\beta := \frac{\hat{\beta}}{2}$, yields a parameterization function for $\rho_{i,j}$ that is consistent with equation 5.3:

$$\rho_{i,j} := \rho(i, j, \hat{\beta}) = \exp\left(-\frac{\hat{\beta}}{2}|i - j|\right), \quad i, j = 1, \ldots, 2M.$$

Practically, it is easier to incorporate this change in parameters directly in the fitting procedure, instead of specifying a transformation of the parameters.

Assuming that the forward rate $F_i(t)$ with annual tenor corresponds to the two semi-annual forward rates $f_{2i-1}(t)$ and f_{2i}, parameterizations are normalized by their tenor $\tau_{i,j} := \tau(T_i, T_j)$. By introducing a mapping of variables

$$f(i) = i \cdot \tau(T_i, T_{i+1}),$$

where τ denotes the tenor of the forward rates, expressed in years, and i is the i-th element in a series of forward rates. $\rho_{i,j}$ then becomes:

$$\rho_{i,j} := \rho(f(i), f(j), f(M)), \quad i, j = 1, \ldots, M.$$

This modification was applied to the full-rank parameterizations from section 4.2 to forward rates of quarterly tenor and semi-annual tenor. The parameters of the tenor-independent parameterizations are given in tables 6.6 to 6.10. Results of applying the tenor-independent approach are given in tables 6.12 and 6.13. 6-month forward rate correlations were obtained from tenor-independent parameterizations derived from 3-month forward rate correlations and vice versa. The tables show the mean square error of the correlation obtained from the change of tenor and of the original parameterized correlation. Errors are between 0.0922% and 0.9661%, which can be considered tolerable with regard to the original parameterization errors that lie within 5.78% and 22.70% (see tables 6.1 to 6.5). Error surfaces are shown in figures 6.23 to 6.27. Errors tend to increase with maturity interval, which can be explained by the fact that the difference in original and modified parameters is amplified with increasing maturity. It should be noted that for those parameterizations that contain a parameter ρ_∞, which denotes (asymptotically) the minimum correlation in the correlation matrix, this interpretation of the parameter no longer holds.

The results show clearly that tenor-independent parameterization does not perform well for change of tenor on historical market correlation. However, for the full-rank parameterizations errors are relatively small, especially when compared to the order of errors between historical market correlation and parameterization in the first place.

Chapter 6

Results

This chapter provides detailed results for the methods presented in this thesis. First, a brief overview of the implementation used for obtaining this chapter's results is given. The remaining sections present the results for calculating historical market correlation, full-rank parameterizations, reduced-rank parameterizations and tenor changes.

6.1 Parameterization implementation

The implementation of the given parameterizations was based on an existing implementation from HfB that provided the Svensson model for market data and some full-rank parameterizations, together with routines for solving unconstrained minimization problems, based on Brent's method and Powell's algorithm as given by (Brent, 2002) and (Press et al., 1992). Some changes to the existing code were made, in terms of adding meaningful constraints, tenor-independence, and newer full-rank parameterization methods. Reduced-rank parameterizations were added, where the eigen decomposition was provided with the code from HfB. The implementation was done in $C\#$.

Powell's algorithm is an unconstrained minimization technique, whereby parameters are fitted over $\mathbb{R}$. As parameters in the parameterizations are constrained, Box-transformations as suggested by (Box, 1966) were employed to map the constrained minimization problem to an unconstrained one:

(i) To make $y \in \mathbb{R}$ fulfil $y \geq c$, the transformation $y := c+|x|$, $x \in \mathbb{R}$ was used. (Box, 1966) suggests the transformation $y := c+x^2$. However, the transformation used is less computationally intensive.

(ii) To make $y \in \mathbb{R}$ fulfil $c_1 \leq y \leq c_2$, the transformation $y := c_1 + (c_2 - c_1) \cdot \sin^2(x)$, $x \in \mathbb{R}$ was used. An alternative transformation, $y := c_1 + (c_2 - c_1) \cdot \frac{e^x}{e^x + e^{-x}}$ circumvents the periodicity introduced by the sine, but comes at a computationally higher cost.

6.2 Historical market correlation

Data used was retrieved from Bloomberg and consists of money market rates of maturities of 1, 3, 6 and 12 months, and swap rates of maturities of $2, \ldots, 10$ years. Daily data from January 1st, 2000, to December 31st, 2004, was used. Day count conventions are $\frac{\text{act}}{360}$ for the money market rates and $\frac{30}{360}$ for the swap rates. A continuous curve respecting the day count conventions was fitted using the model described by (Svensson, 1994). The model allows for retrieving forward rates of arbitrary tenor and maturity. Tenors used for the parameterizations were 3 months and 6 months.
Data was aggregated into yearly sets and for each data set a series of adjacent forward rates was calculated. Here, the maturity of each forward was fixed and the time-to-maturity decreased while moving through time. 40 quarterly forward rates and 20 semi-annual forward rates were calculated for each daily sample. For each year, a 40×40 correlation matrix for quarterly forward rates and a 20×20 correlation matrix for semi-annual rates were calculated. Chapter 3 refers in detail to Svensson's model, calculation of forward rates and calculation of correlation. The unparameterized correlation is referred to as historical market correlation.
The historical market correlation of forward rates of 3-month tenor and 6-month tenor as estimated from market data is presented in figures 6.1 and 6.2, respectively.

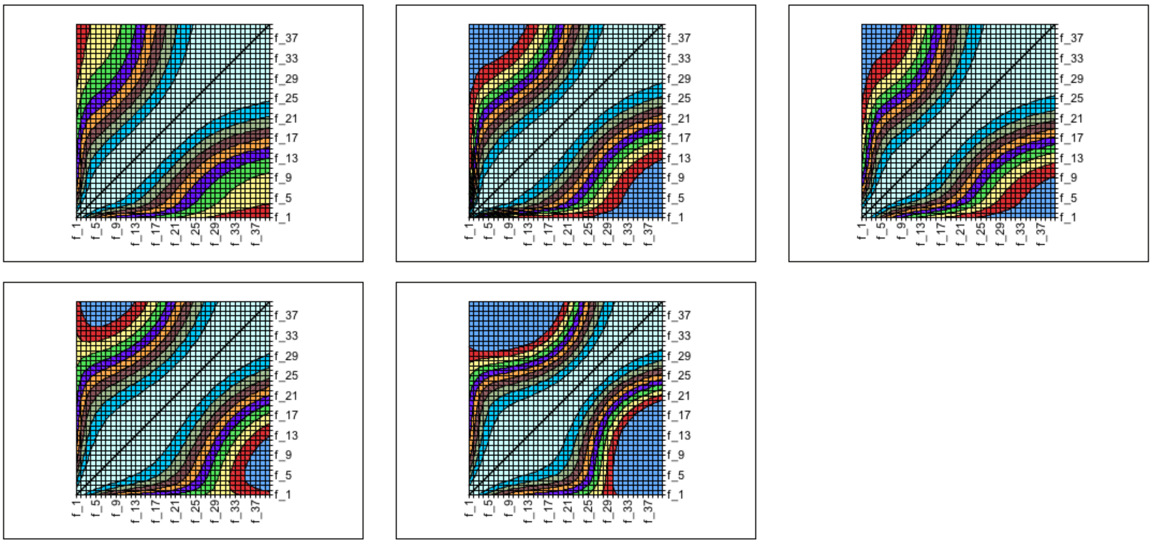


Figure 6.1: Historical market correlation of 3-month forward rates. Top left to bottom right: 2000, 2001, 2002, 2003, 2004.

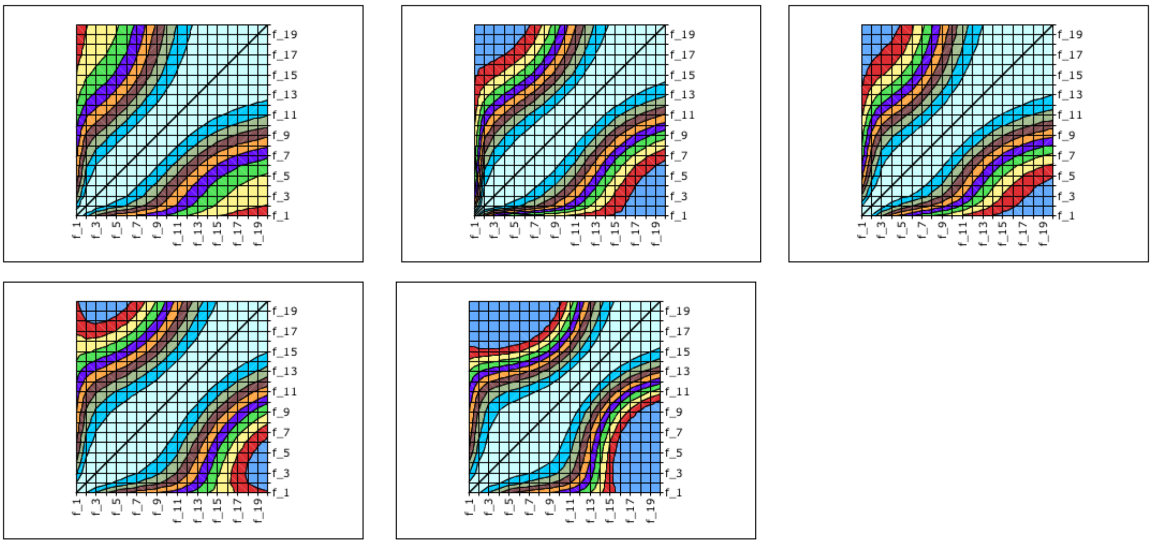
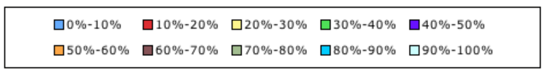


Figure 6.2: Historical market correlation of 6-month forward rates. Top left to bottom right: 2000, 2001, 2002, 2003, 2004.



6.3 Full-rank parameterizations

Results from the following full-rank parameterizations for all correlation matrices, i.e. 3-month tenor and 6-month tenor, years $2000, \ldots, 2004$, are presented here. The parameterizations are:

(i) One-parameter parameterization from definition 4.3,

(ii) Improved two-parameters parameterization from definition 4.6,

(iii) Classical two-parameter parameterization from definition 4.7,

(iv) Three-parameter parameterization from definition 4.8 and

(v) Two-parameter, square-root parameterization from definition 4.9.

The parameters as obtained from least-squares minimization are given in tables 6.1 to 6.5.

The correlation matrices are outlined for each year and each tenor in figures 6.3 to 6.12.

Tenor		2000	2001	2002	2003	2004
3m	β	0.0334	0.0481	0.0428	0.0394	0.05875
	MSE	12.70%	16.83%	15.84%	13.39%	21.94%
6m	β	0.0663	0.0956	0.0853	0.0783	0.1171
	MSE	12.33%	16.16%	15.62%	13.20%	21.94%

Table 6.1: One parameter parameterization, see equation (4.3).

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	0.2392	0.1345	0.1587	0.2152	0.1011
	η	0.8838	0.8561	1.1082	0	0
	MSE	9.36%	15.32%	12.80%	13.39%	21.94%
6m	ρ_∞	0.2538	0.1455	0.1697	0.2557	0.1081
	η	0.7802	0.7375	0.9914	0	0
	MSE	9.21%	14.79%	12.75%	13.20%	21.94%

Table 6.2: Improved two-parameters, stable parameterization, see equation (4.6).

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-0.9995	-0.9993	-0.9998	-1	-1
	β	0.0142	0.0193	0.0177	0.0165	0.0232
	MSE	11.39%	14.32%	13.48%	10.87%	17.66%
6m	ρ_∞	-1	-1	-1	-0.7075	-1
	β	0.0282	0.0384	0.0351	0.0398	0.0464
	MSE	10.98%	13.52%	13.23%	10.99%	17.65%

Table 6.3: Classical two-parameters parameterization, see equation (4.8).

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-0.0976	-0.4502	-0.3655	-0.9984	-0.9984
	β	0.0531	0.0427	0.0481	0.0232	0.0232
	α	0.0591	0.0385	0.0536	0	0
	MSE	6.07%	11.73%	7.71%	10.40%	17.67%
6m	ρ_∞	-0.1269	-0.5735	-0.4132	-0.9998	-0.9987
	β	0.1020	0.0738	0.0920	0.0367	0.0464
	α	0.1133	0.0678	0.1034	0.0220	0
	MSE	5.78%	10.98%	7.58%	10.21%	17.65%

Table 6.4: Three-parameters parameterization, see equation (4.10).

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-0.9997	-1	-0.9997	-0.9881	-0.9125
	β	0.1170	0.1582	0.1456	0.1318	0.1928
	MSE	7.97%	12.31%	10.09%	13.25%	22.99%
6m	ρ_∞	-1	-0.9992	-0.9997	-0.9962	-0.9951
	β	0.1679	0.2271	0.2090	0.1890	0.2645
	MSE	7.87%	11.84%	10.07%	13.05%	22.70%

Table 6.5: Two-parameters, square-root parameterization, see equation (4.12).

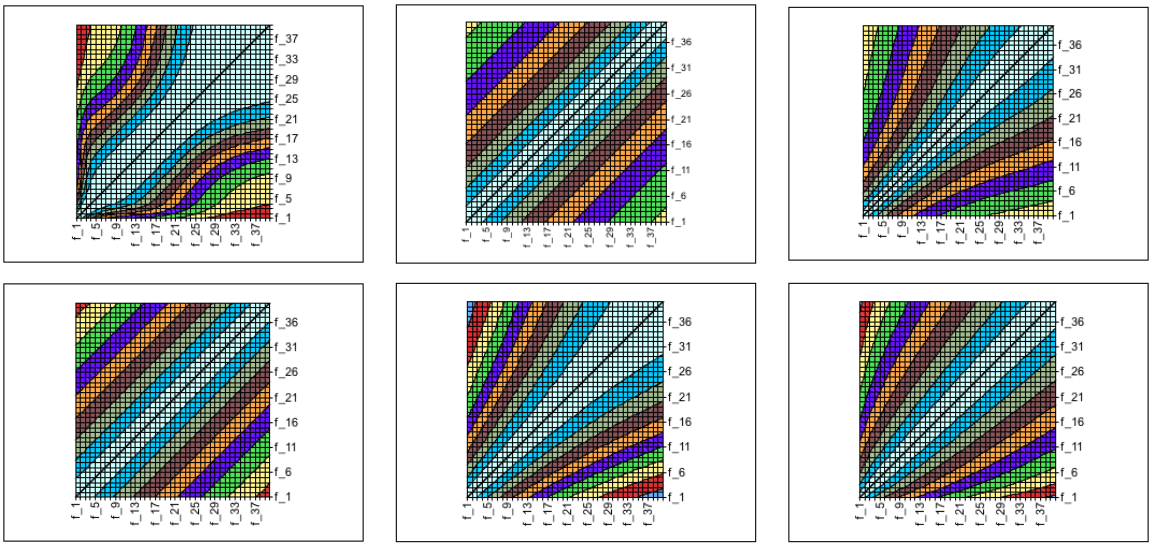


Figure 6.3: 3-month forward rate full-rank correlation matrices, data of 2000. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

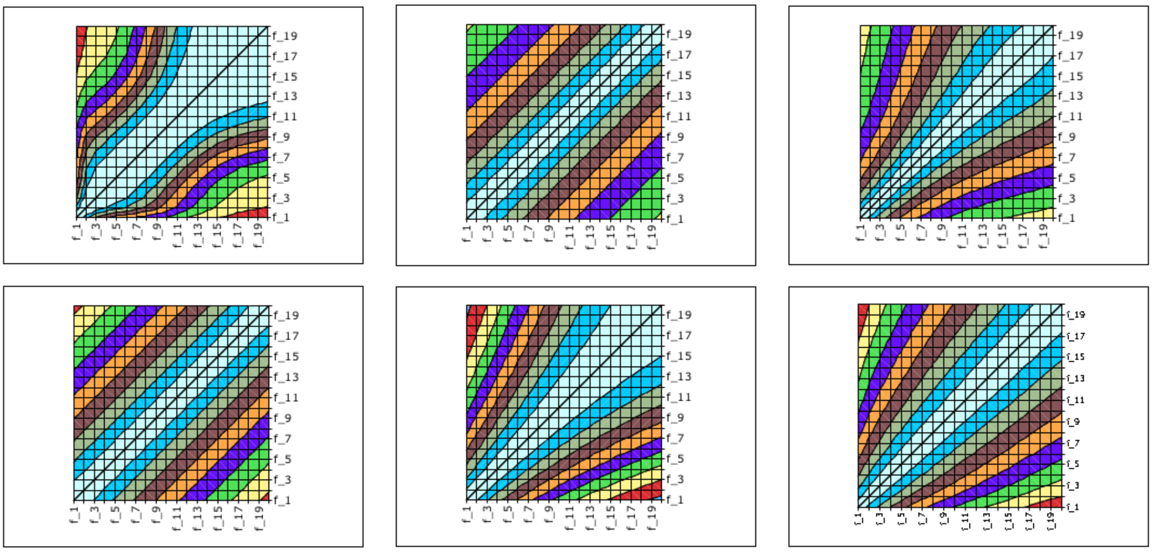
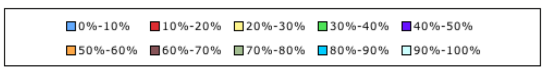


Figure 6.4: 6-month forward rate full-rank correlation matrices, data of 2000. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.



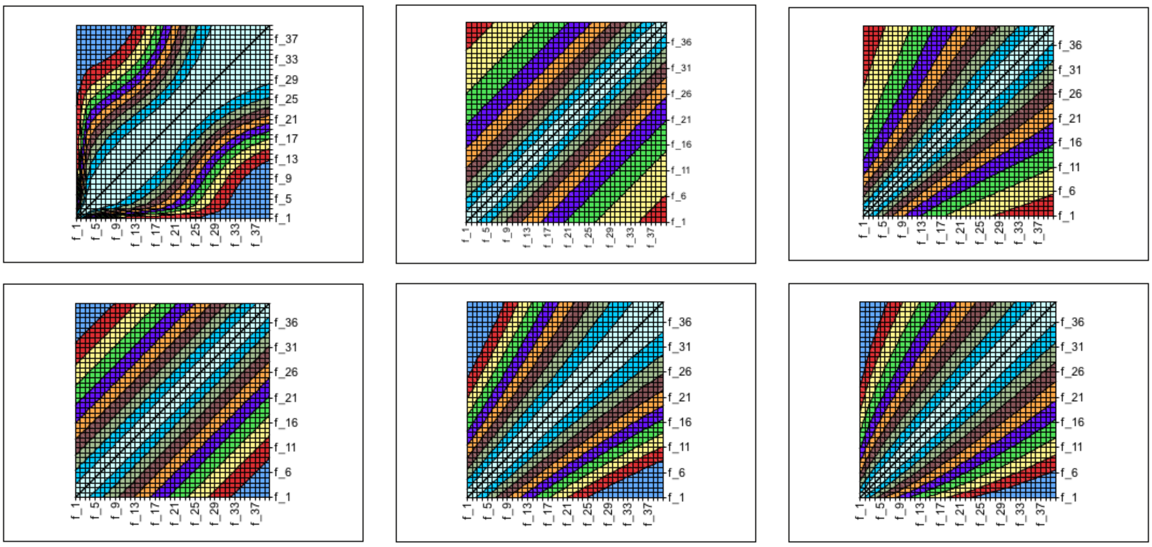


Figure 6.5: 3-month forward rate full-rank correlation matrices, data of 2001. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

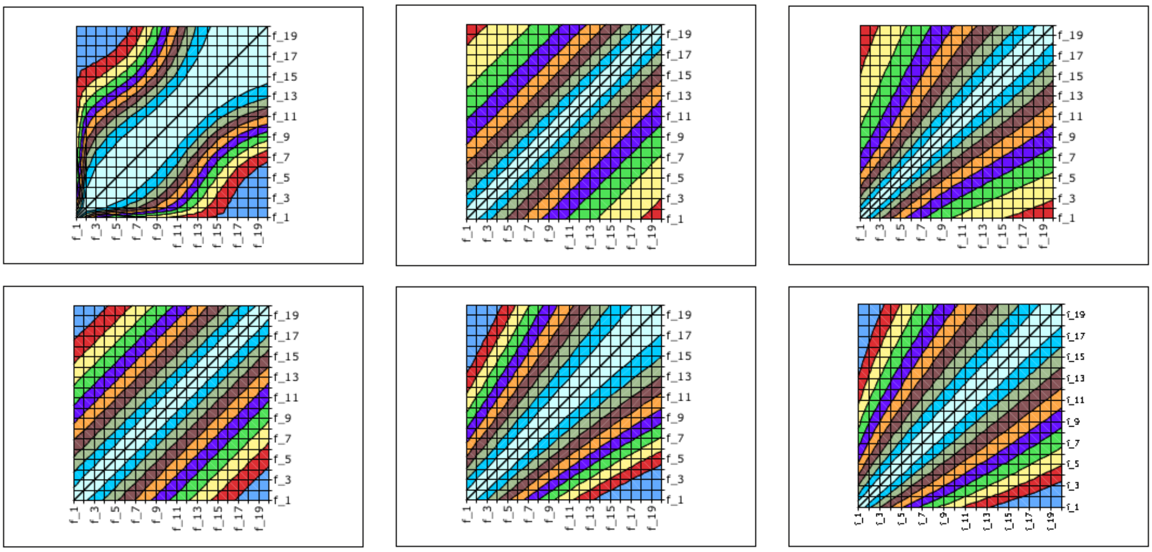
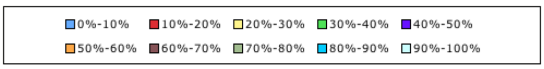


Figure 6.6: 6-month forward rate full-rank correlation matrices, data of 2001. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.



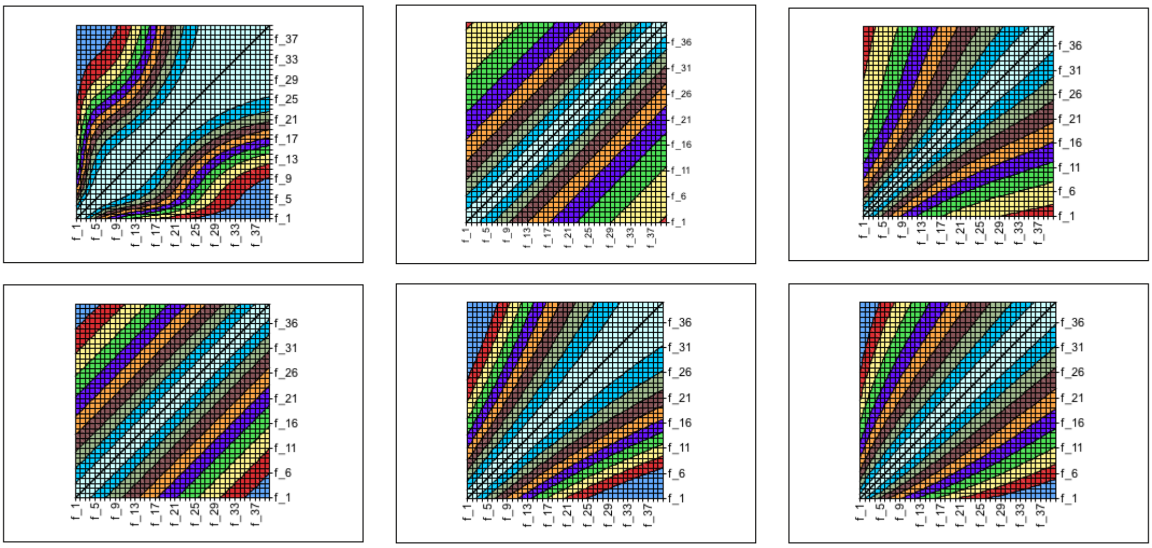


Figure 6.7: 3-month forward rate full-rank correlation matrices, data of 2002. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

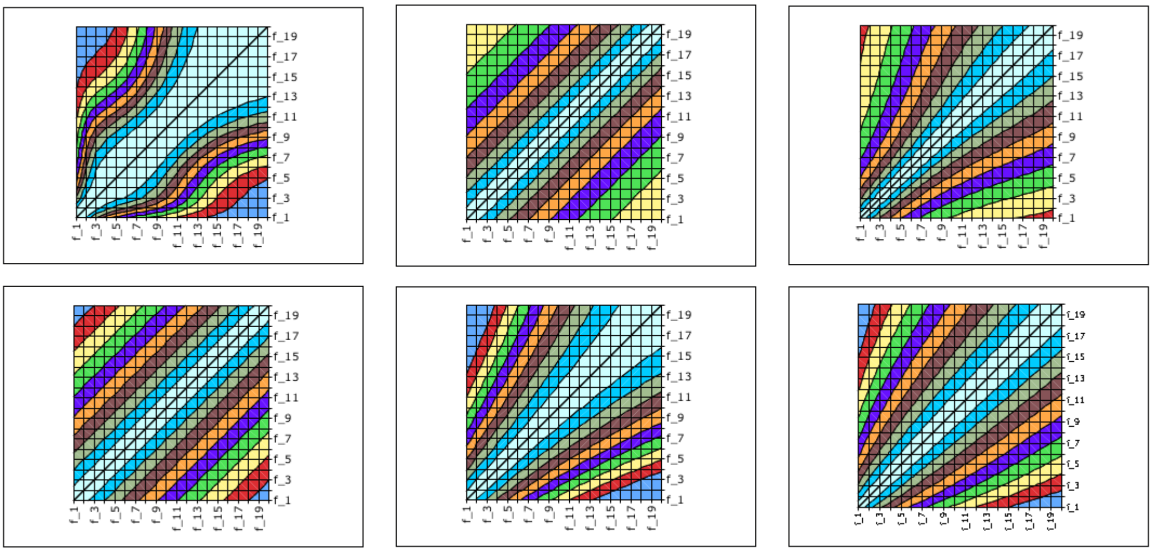


Figure 6.8: 6-month forward rate full-rank correlation matrices, data of 2002. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

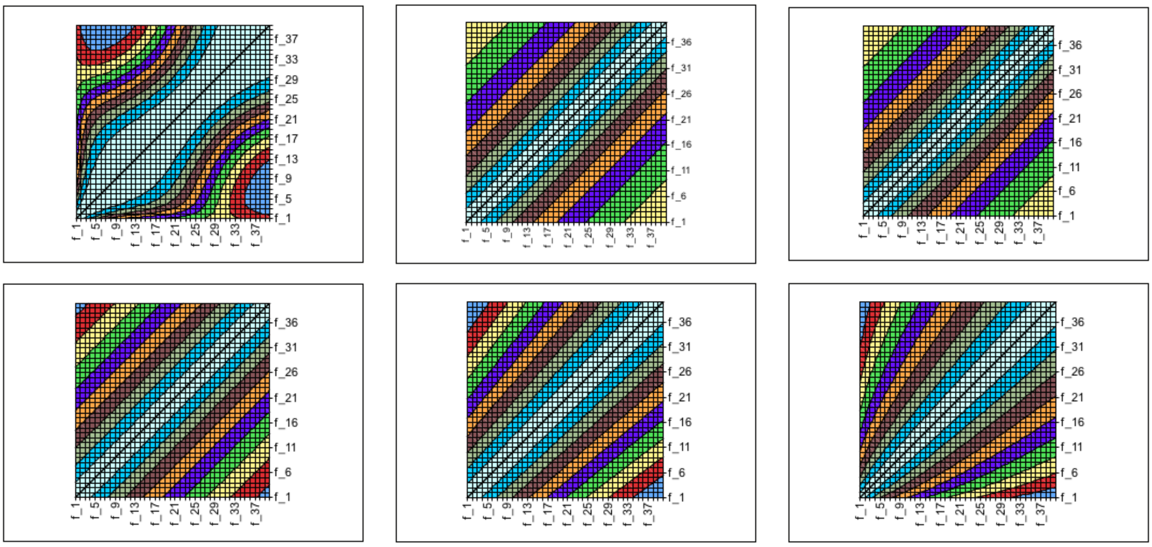


Figure 6.9: 3-month forward rate full-rank correlation matrices, data of 2003. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

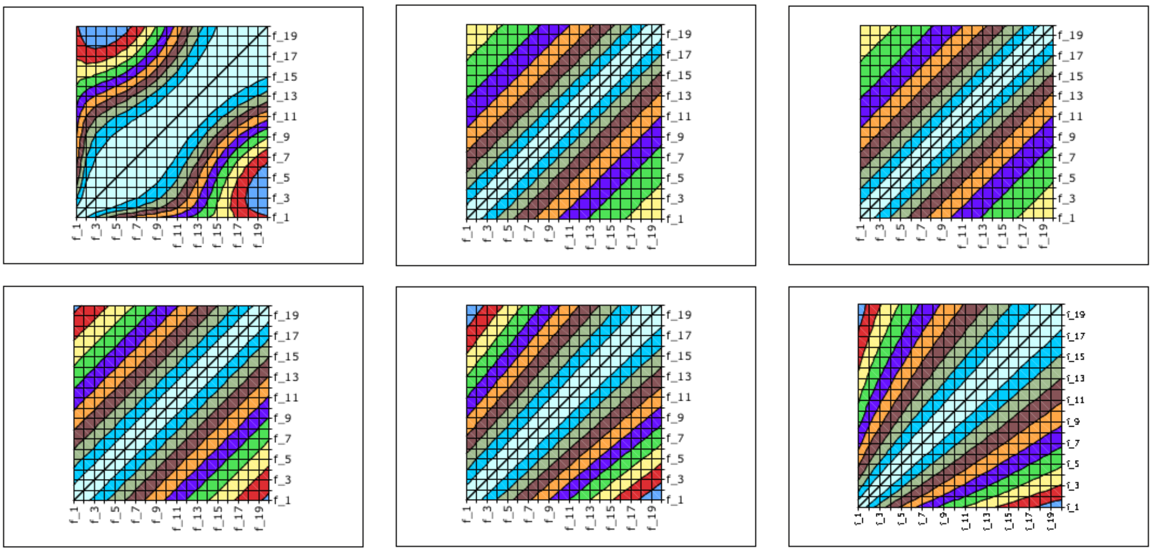
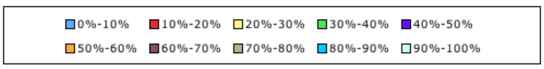


Figure 6.10: 6-month forward rate full-rank correlation matrices, data of 2003. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.



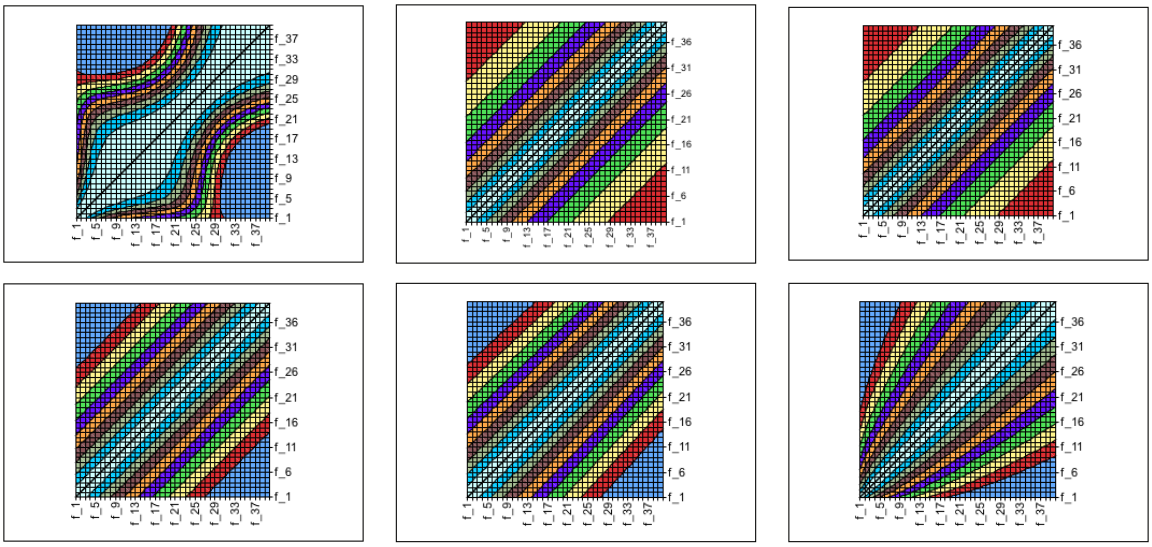


Figure 6.11: 3-month forward rate full-rank correlation matrices, data of 2004. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

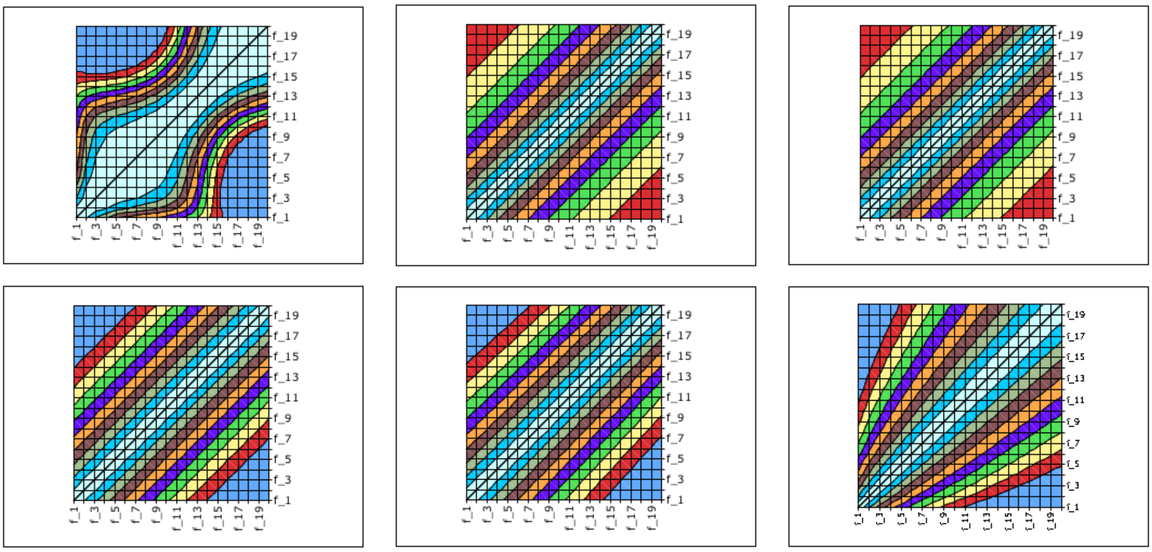


Figure 6.12: 6-month forward rate full-rank correlation matrices, data of 2004. Top left to bottom right: Historical market data; One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

6.4 Reduced-rank parameterizations

Section 4.4 presents two methods for fitting the market model to fewer independent Brownian motions than forward rates in the model. This is achieved through reduced-rank correlation matrices. The first approach uses principal component analysis to reduce the number of significant eigenvectors and eigenvalues from maximum M to 3. The second approach reduces the rank through a hypersphere decomposition as given by equation (4.13). The correlation matrices as obtained by principal component analysis and hypersphere decomposition, as well as the three most significant eigenvectors of the principals component analysis are given in figures 6.13 to 6.21.

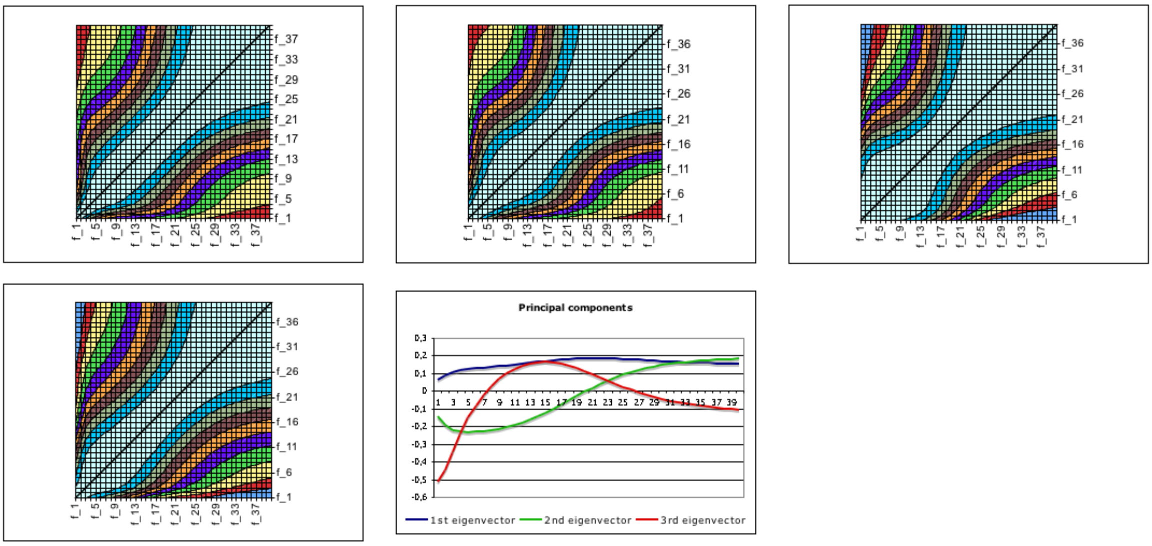


Figure 6.13: 3-month forward rate reduced-rank correlation matrices, data of 2000. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

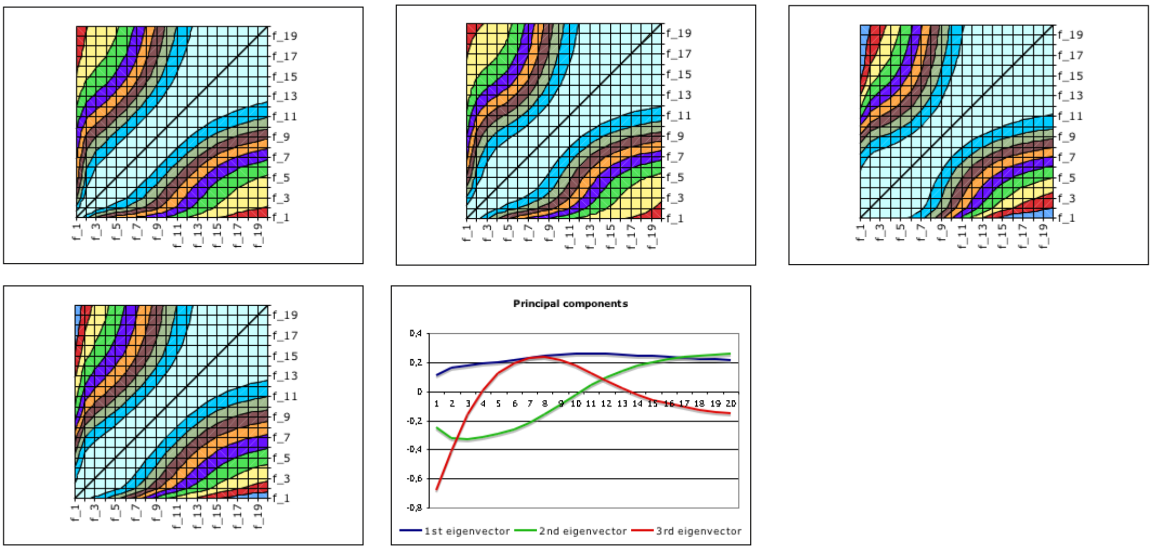


Figure 6.14: 6-month forward rate reduced-rank correlation matrices, data of 2000. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

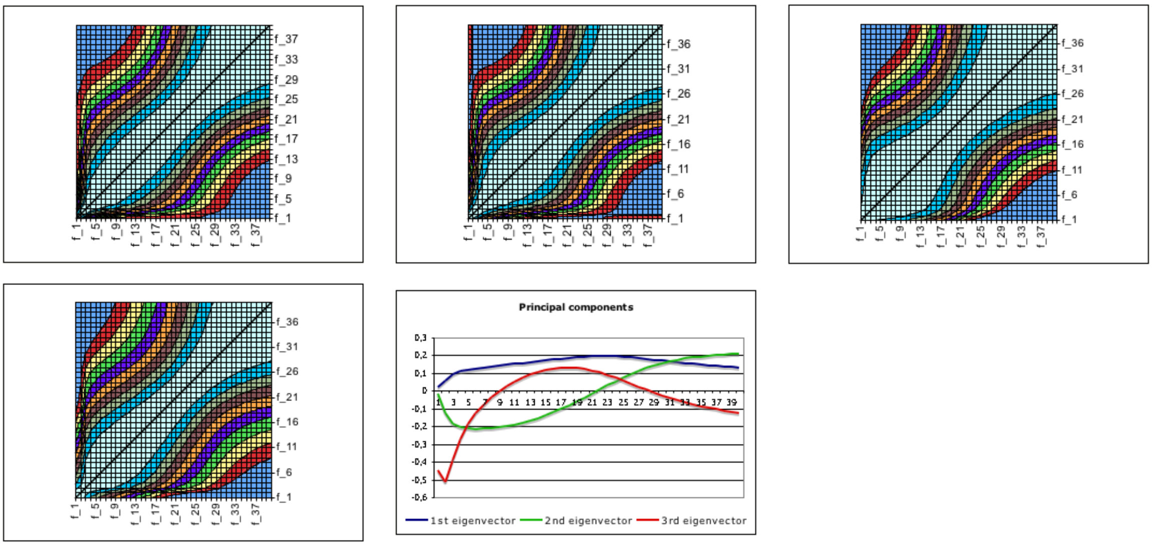


Figure 6.15: 3-month forward rate reduced-rank correlation matrices, data of 2001. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

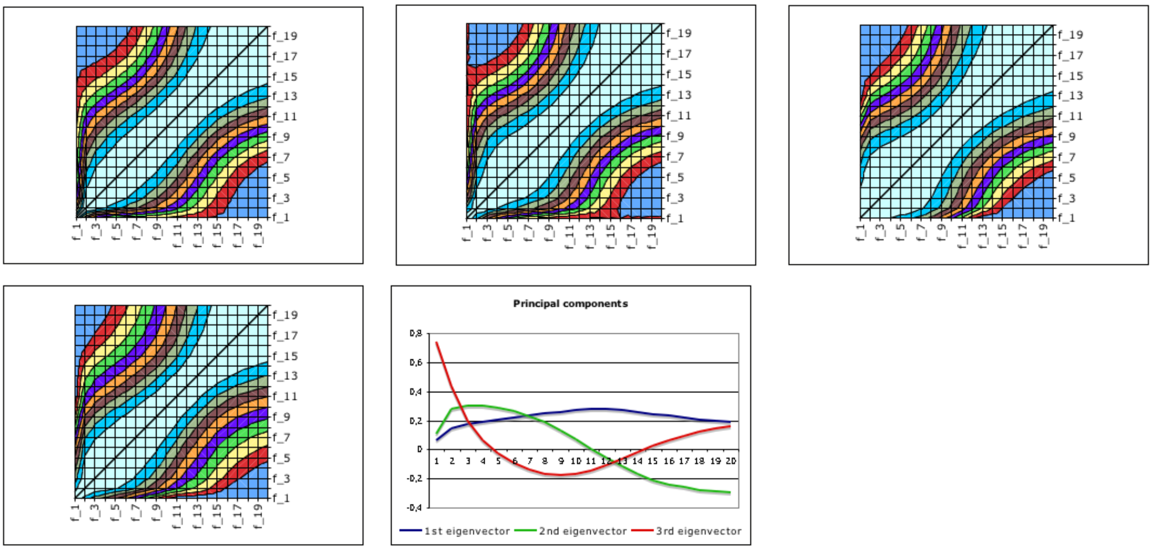
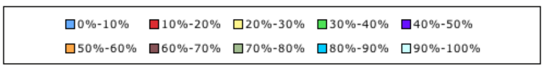


Figure 6.16: 6-month forward rate reduced-rank correlation matrices, data of 2001. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.



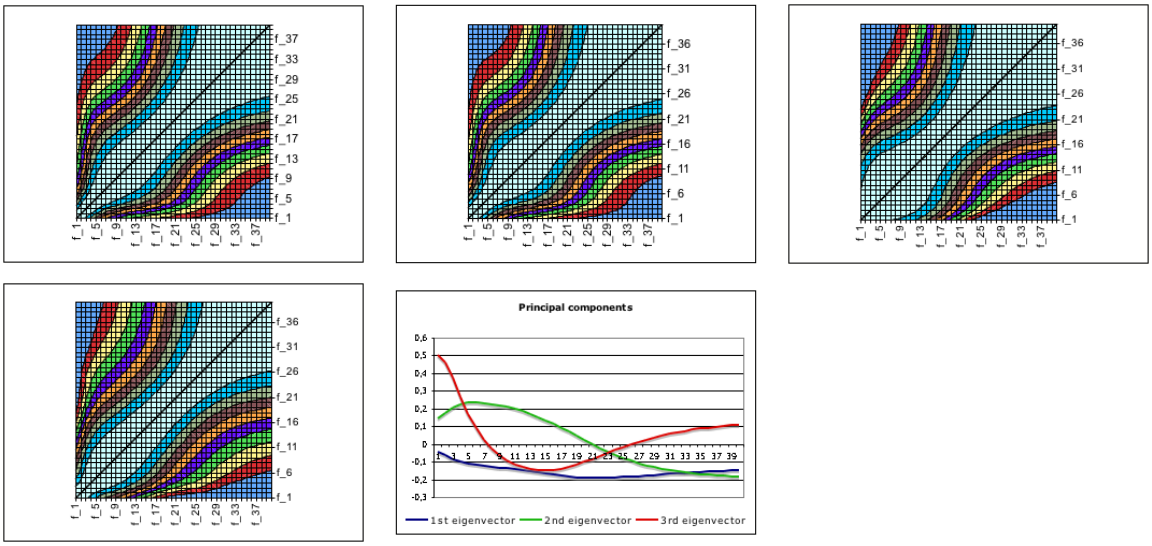


Figure 6.17: 3-month forward rate reduced-rank correlation matrices, data of 2002. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

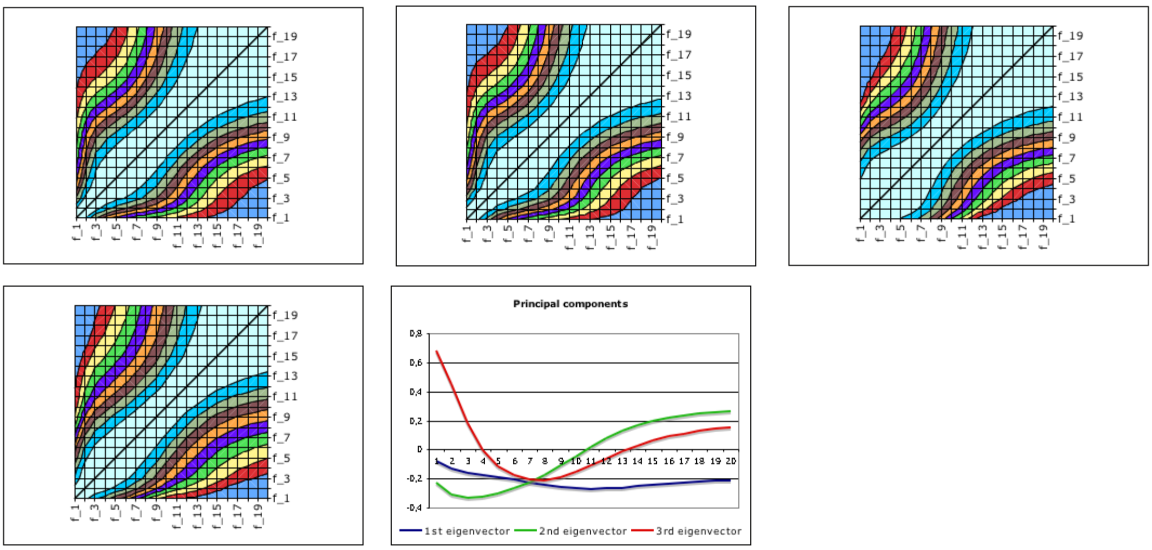
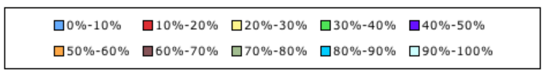


Figure 6.18: 6-month forward rate reduced-rank correlation matrices, data of 2002. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.



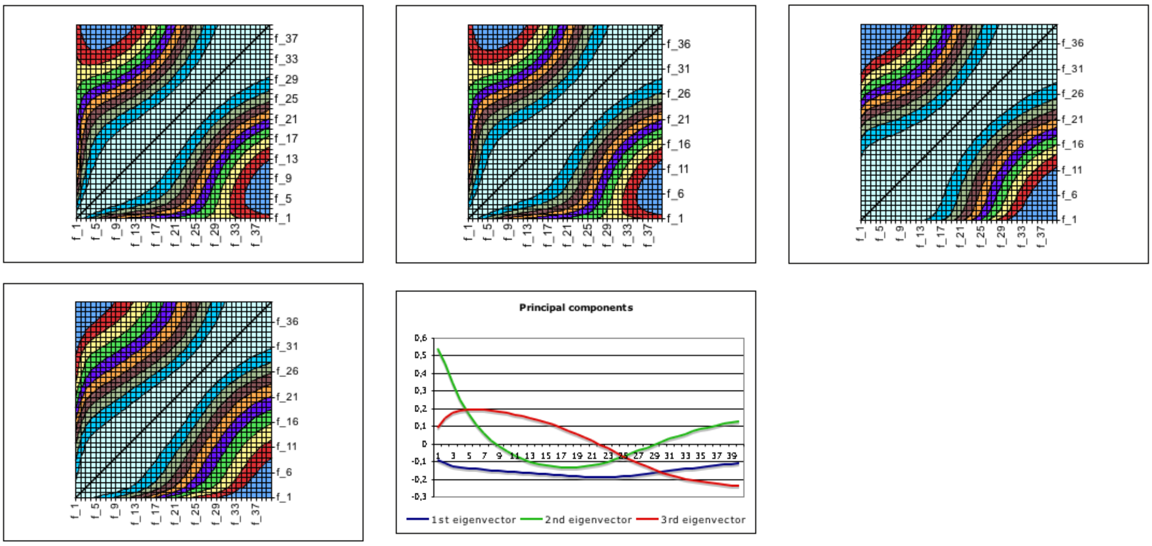


Figure 6.19: 3-month forward rate reduced-rank correlation matrices, data of 2003. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

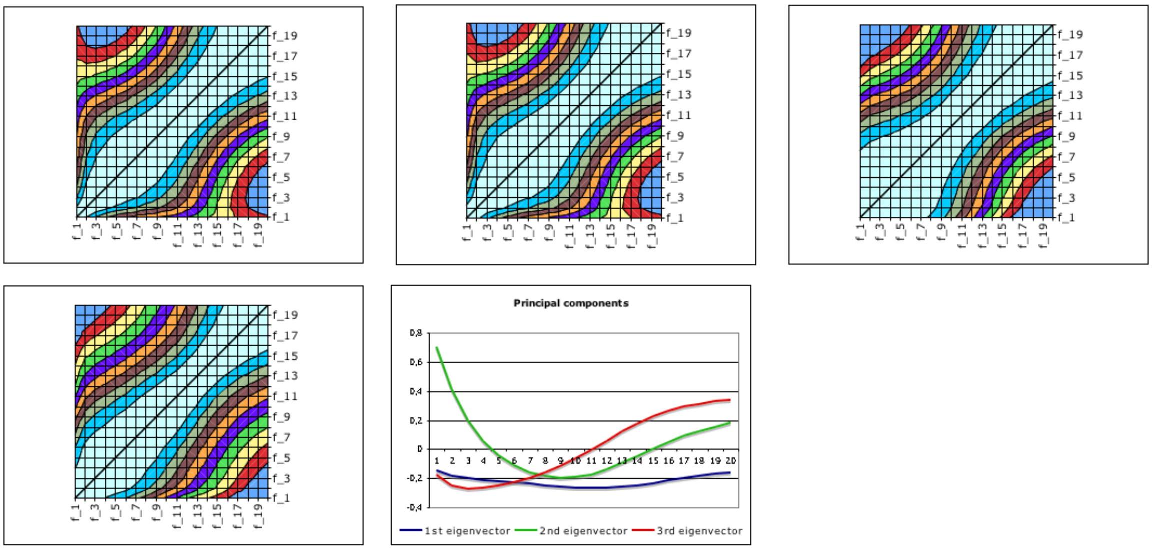


Figure 6.20: 6-month forward rate reduced-rank correlation matrices, data of 2003. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

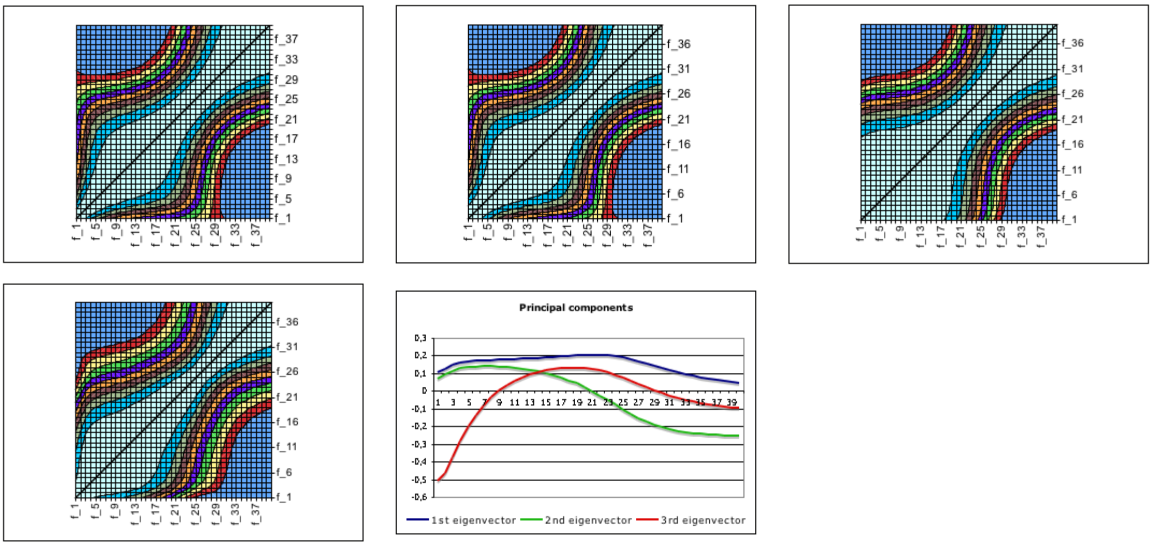


Figure 6.21: 3-month forward rate reduced-rank correlation matrices, data of 2004. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

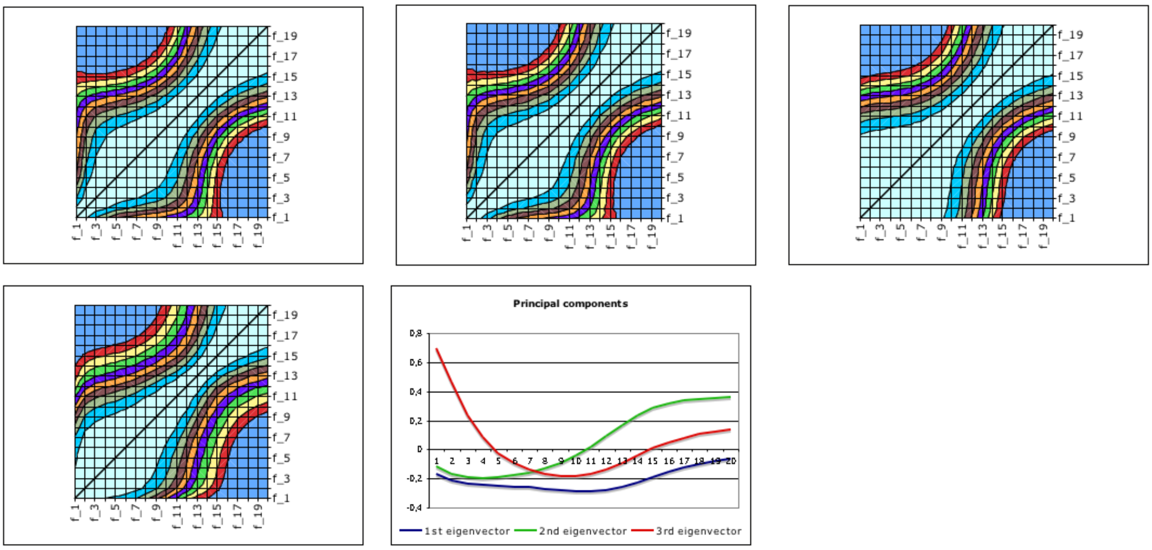


Figure 6.22: 6-month forward rate reduced-rank correlation matrices, data of 2004. Top left to bottom right: Historical market data; Principal component analysis of rank 3; Hypersphere decomposition (rank 2); Hypersphere decomposition (rank 4); Three most significant eigenvectors of principal component analysis.

6.5 Tenor changes

Two types of tenor changes were presented in chapter 5, the approximation given by (Alexander, 2003), and an interpolation to make parameters tenor-independent. Both methods were applied to all full-rank parameterizations.
The tenor-independent parameterizations are given in tables 6.6 to 6.10. Mean square errors obtained by applying the tenor changes are given in tables 6.11, 6.12 and 6.13. Error surfaces of the absolute error between the result of the tenor-independent parameterization of 6-month forward rate correlations, applied to 3-month forward rates, and the corresponding 3-month forward parameterizations are shown in figures 6.23 to 6.27.

Tenor		2000	2001	2002	2003	2004
3m	β	0.1336	0.1923	0.1713	0.1576	0.2350
	MSE	12.70%	16.83%	15.84%	13.39%	21.94%
6m	β	0.1326	0.1910	0.1705	0.1567	0.2342
	MSE	12.33%	16.16%	15.62%	13.20%	21.94%

Table 6.6: One parameter parameterization, see equation (4.3), parameters are tenor-independent.

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	0.2995	0.1755	0.2112	0.2423	0.1206
	β	0.6143	0.5951	0.7697	0	0
	MSE	9.43%	15.33%	12.86%	13.39%	21.94%
6m	ρ_∞	0.2935	0.1729	0.2050	0.2441	0.1215
	β	0.6071	0.5732	0.7707	0	0
	MSE	9.26%	14.80%	12.80%	13.20%	21.94%

Table 6.7: Improved two-parameters, stable parameterization, see equation (4.6), parameters are tenor-independent.

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-0.9997	-0.9998	-1	-1	-1
	β	0.0567	0.0771	0.0708	0.0660	0.0930
	MSE	11.39%	14.32%	13.48%	10.86%	17.66%
6m	ρ_∞	-0.9997	-0.9999	-1	-1	-1
	β	0.0563	0.0768	0.0703	0.0657	0.09280
	MSE	10.98%	13.52%	13.23%	10.63%	17.65%

Table 6.8: Classical two-parameters parameterization, see equation (4.8), parameters are tenor-independent.

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-0.1165	-0.4742	-0.3749	-1	-1
	β	0.2070	0.1646	0.1905	0.0736	0.0901
	α	0.2370	0.1473	0.2141	0.0467	-0.0123
	MSE	6.07%	11.73%	7.71%	10.40%	17.63%
6m	ρ_∞	-0.1291	-0.5838	-0.4123	-1	-1
	β	0.2030	0.1462	0.1842	0.0735	0.0894
	α	0.2254	0.1348	0.20671	0.0450	-0.0137
	MSE	5.78%	10.98%	7.58%	10.21%	17.65%

Table 6.9: Three-parameters parameterization, see equation (4.10), parameters are tenor-independent.

Tenor		2000	2001	2002	2003	2004
3m	ρ_∞	-01	-1	-1	-1	-1
	β	0.2339	0.3165	0.2906	0.2624	0.3656
	MSE	7.97%	12.31%	10.09%	13.25%	22.98%
6m	ρ_∞	-1	-1	-1	-1	-1
	β	0.2375	0.3215	0.2959	0.1567	0.3737
	MSE	7.87%	11.84%	10.07%	13.20%	22.70%

Table 6.10: Two-parameters, square-root parameterization, see equation (4.12), parameters are tenor-independent.

Method	Mean-square error				
	2000	2001	2002	2003	2004
Historical market correlation	0.1823%	0.1379%	0.1518%	0.3202%	0.2048%
One-parameter	0.9792%	1.2892%	1.2315%	1.1393%	1.5443%
Improved two-parameters	1.0809%	1.3701%	1.3396%	1.1393%	1.5443%
Classical two-parameters	0.8388%	1.0651%	1.0288%	0.9692%	1.2559%
Three-parameters	0.7453%	0.8834%	0.7525%	0.9012%	1.3339%
Two-parameters, square-root	0.9381%	1.1702%	1.1549%	1.1058%	1.4238%

Table 6.11: Mean-square error of approximation of (Alexander, 2003) of 6-month forward rate correlations from 3-month forward rate correlations against parameterized 6-month forward rate correlations.

Method	Mean-square error				
	2000	2001	2002	2003	2004
Historical market correlation	1.6304%	3.1360%	1.5359%	1.5916%	2.0246%
One-parameter	0.1929%	0.1926%	0.1311%	0.1582%	0.1034%
Improved two-parameters	0.4933%	0.3989%	0.5574%	0.1582%	0.1034%
Classical two-parameters	0.1921%	0.1826%	0.1287%	0.1485%	0.0922%
Three-parameters	0.6796%	0.7093%	0.7523%	0.1125%	0.2004%
Two-parameters, square-root	0.4849%	0.6073%	0.6676%	0.6674%	0.9093%

Table 6.12: Mean-square error of estimation of 6-month forward rate correlations from 3-month forward rate correlations by using tenor-independent parameters.

Method	Mean-square error				
	2000	2001	2002	2003	2004
One-parameter	0.1930%	0.1926%	0.1311%	0.1583%	0.1035%
Improved two-parameters	0.4808%	0.3877%	0.5488%	0.1583%	0.1035%
Classical two-parameters	0.1922%	0.1827%	0.1288%	0.1486%	0.0922%
Three-parameters	0.6628%	0.7039%	0.7458%	0.1100%	0.2070%
Two-parameters, square-root	0.5214%	0.6841%	0.7140%	0.7156%	0.9661%

Table 6.13: Mean-square error of estimation of 3-month forward rate correlations from 6-month forward rate correlations by using tenor-independent parameters.

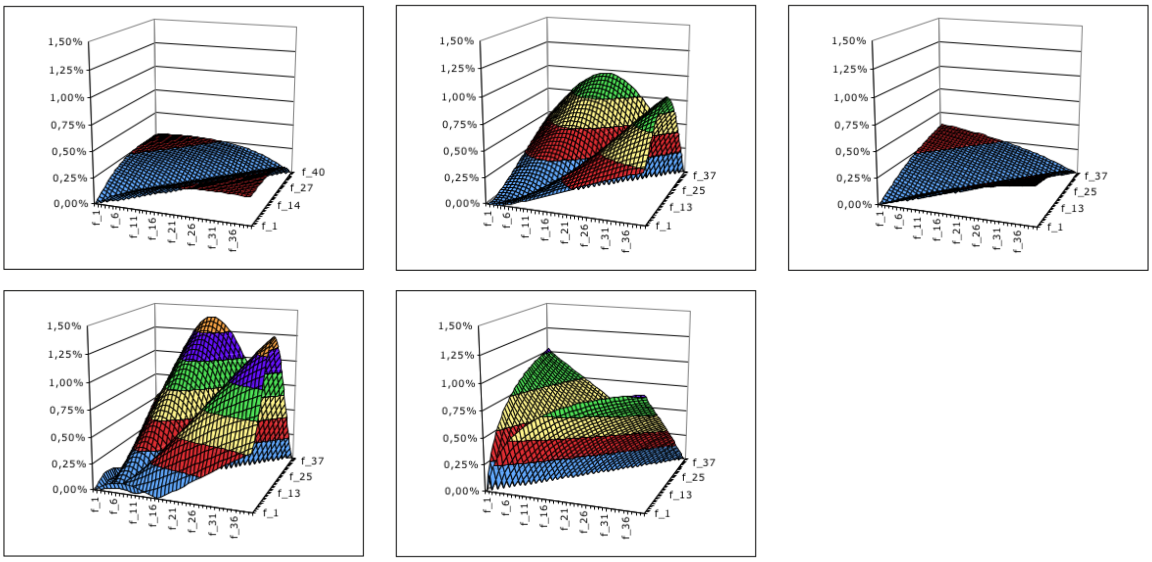


Figure 6.23: Error surfaces of tenor-change applied to full-rank matrices from 6-month to 3-month forward rates, data of 2000. Top left to bottom right: One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

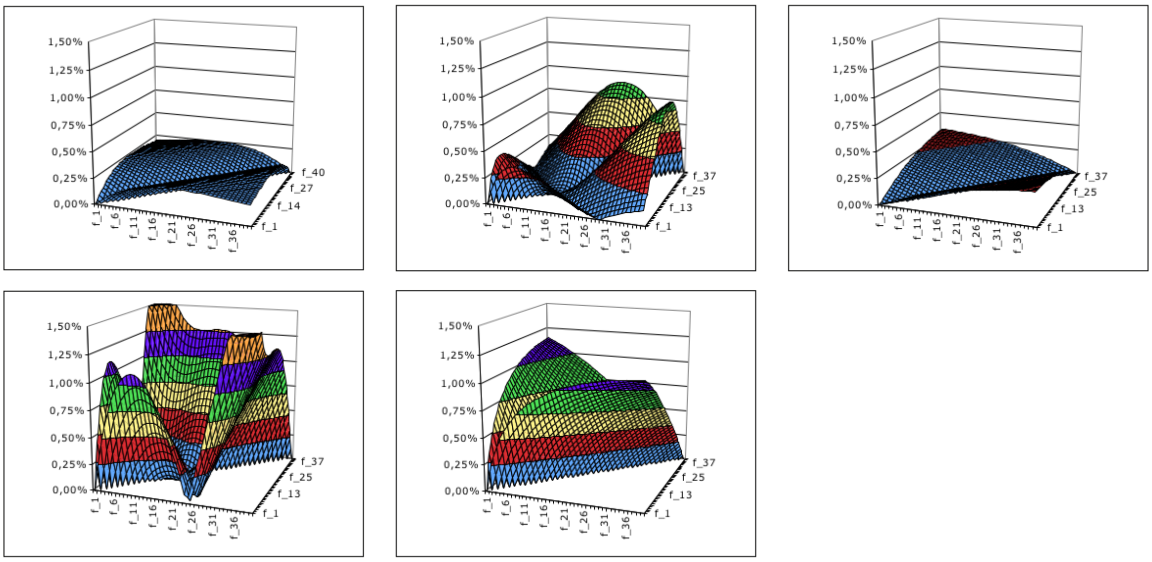


Figure 6.24: Error surfaces of tenor-change applied to full-rank matrices from 6-month to 3-month forward rates, data of 2001. Top left to bottom right: One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

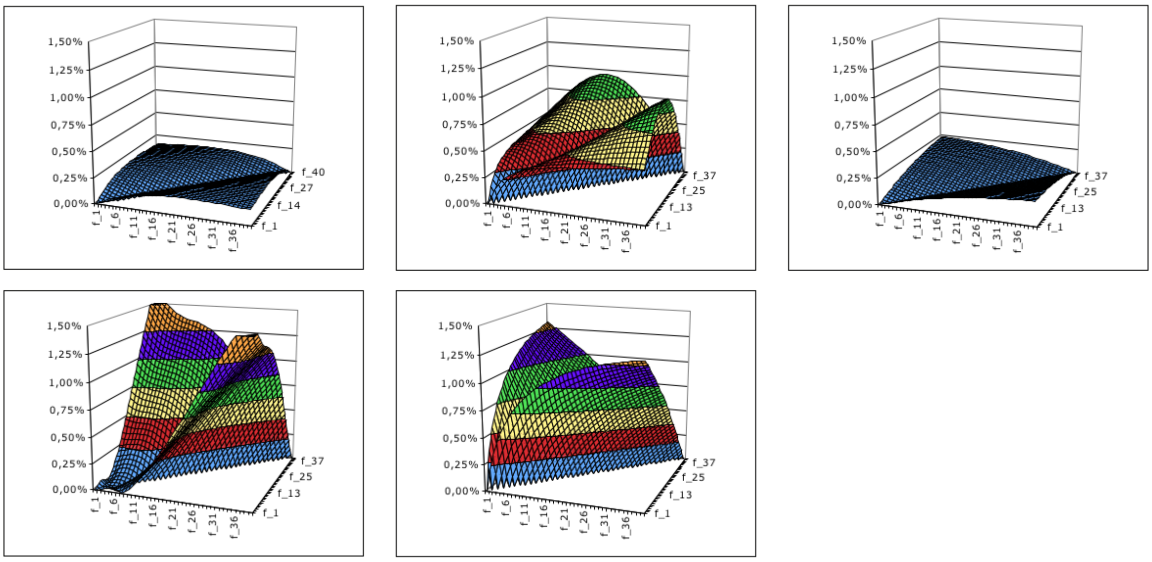


Figure 6.25: Error surfaces of tenor-change applied to full-rank matrices from 6-month to 3-month forward rates, data of 2002. Top left to bottom right: One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

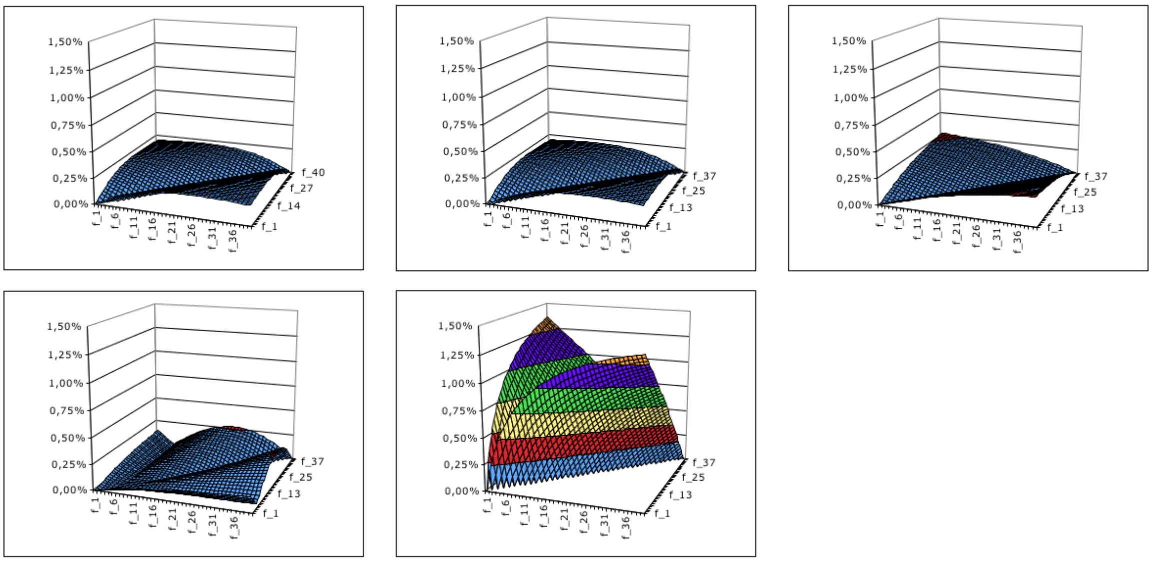


Figure 6.26: Error surfaces of tenor-change applied to full-rank matrices from 6-month to 3-month forward rates, data of 2003. Top left to bottom right: One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

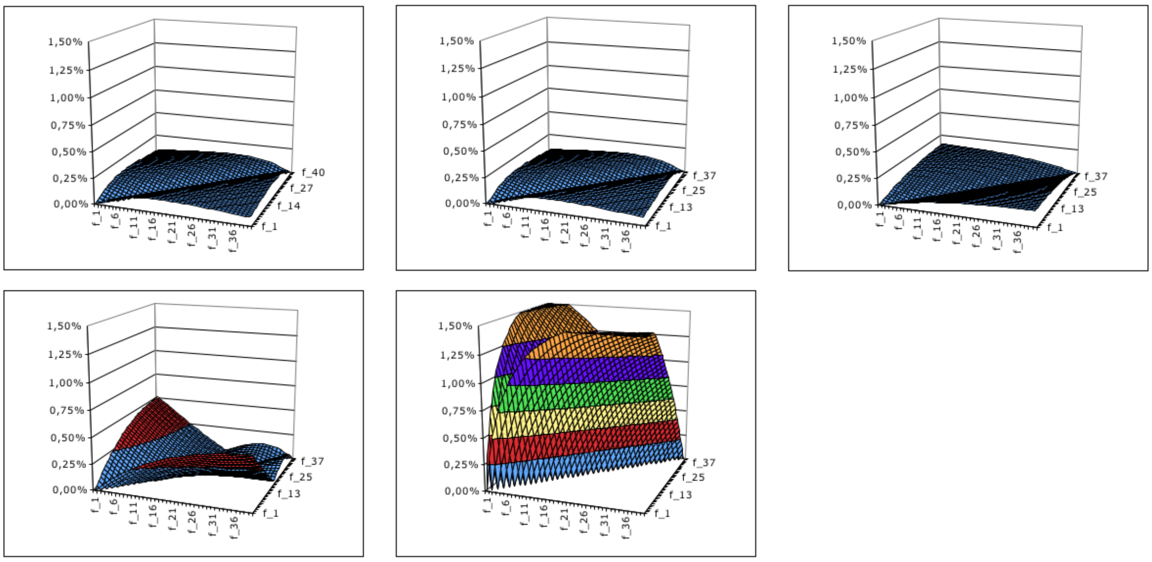


Figure 6.27: Error surfaces of tenor-change applied to full-rank matrices from 6-month to 3-month forward rates, data of 2004. Top left to bottom right: One-parameter parameterization; Improved two-parameter parameterization; Classical two-parameter parameterization; Three-parameter parameterization; Two-parameter, square-root parameterization.

Chapter 7

Conclusion

The LIBOR market model is a market model that is calibrated to reproduce market-quoted cap volatilities. Some path-dependent financial instruments that span several caplet maturities are sensitive to correlation of caplet volatilities. To price these products with a LIBOR market model, the model must additionally be calibrated to allow for correlation among the caplets. Forward rates being the underlying of caplets, the LIBOR market model is made up of a set of SDEs that represent the evolution of forward rates with time. In the LIBOR market model, instantaneous correlation, i.e. the correlation among time-increments of the forward rates, is modelled. However, terminal correlation, which depends on both instantaneous volatility and instantaneous correlation, is the correlation that can be observed for market-quoted correlation-sensitive instruments.

Correlation is usually either obtained from correlation-sensitive market-quoted products, such as European swaptions, or from historical market data, such as forward rates. Since there is no market-quoted instrument that depends solely on instantaneous correlation, it is a common approach to calculate instantaneous correlation from historical data.

In this thesis, historical money market rates and swap rates of maturities up to 10 years of the years from 2000 to 2004 were fitted to (Svensson, 1994)'s model for parameterizing yield curves. The model allows for retrieving forward rates of arbitrary maturity and tenor. For each year, adjacent series of lognormal forward rate returns of 3-month tenor and 6-month tenor were retrieved. Instantaneous correlation among the adjacent series was calculated, yielding a 40×40 correlation matrix in the case of 3-month forward rates and a 20×20 correlation matrix in the case of 6-month forward

rates.
Some parameterizations for the correlation matrix have been developed over the past few years. They can be found for example in (Schoenmakers and Coffey, 2000), (Brigo, 2002), (Rebonato, 2004). There are several advantages to fitting the correlation matrix obtained from market data to a functional form: the rank, i.e. the number of driving factors of the model, can be explicitly chosen, and general properties of the correlation matrix can be enforced. Two such properties are widely accepted, namely that decorrelation increases with increasing maturity interval between two forward rates, and the correlation increases for increasing maturities, while keeping the maturity interval constant.
Some full-rank and reduced-rank parameterizations from the above references are presented and analysed. The historical market data was applied to the data and results for the parameterizations are given.
Full-rank parameterizations typically have between one and three parameters. They do not reproduce market data very accurately, but serve to model desired properties and to eliminate properties that are specific to a given data set.
On the other hand, reduced-rank parameterizations, such as principal component analysis and hypersphere decomposition, have a higher number of parameters and therefore provide a more precise fit to the given market correlation.
Given a parameterized correlation matrix that is fit to forward rates of a particular tenor, it may be desirable to retrieve correlation of forward rates of different tenor. Consider as an example a LIBOR market model that has been calibrated to caplets of 6-month tenor, but that is to be used to price an instrument of 1-year tenor. (Brigo and Mercurio, 2001) and (Alexander, 2003) present such an approach where the SDE of a forward rate of annual tenor is modelled by the stochastic processes of the two semi-annual forward rates spanning the annual rate. This approach was applied to the historical market correlation and the parameterizations to obtain correlation of 6-month forward rates from 3-month forward rates.
An alternative approach developed here can be applied to retrieve correlation of reduced tenor. It is shown that the error between full-rank parameterized correlation matrices of different tenors is small, which can be attributed to the fact that the general properties enforced by the full-rank parameterizations are independent of the tenor of the underlying rates. The inherent idea is therefore that a given correlation matrix can be interpolated to deliver correlation for rates of arbitrary tenor. The

tenor-independent approach is therefore well-suited for application to full-rank parameterized correlation. It does not lend itself for application to historical market correlation or to reduced-rank parameterizations. Results on the historical market correlation and full-rank parameterizations are provided for obtaining correlation of 3-month forward rates given correlation of 6-month forward rates, and vice versa.

Appendix A

Stochastic calculus

The valuation of financial derivatives is driven by the fundamental assumption of absence of arbitrage of financial markets: two portfolios with identical payoffs must have the same price. This argument is particularly useful for devising hedging strategies and it is with this argument that the famous option pricing formula by (Black and Scholes, 1973) was derived.
The basic building block of no-arbitrage theory is to model the evolution of financial assets in time with stochastic processes or stochastic differential equations (SDEs). An important probability measure associated with an SDE is the martingale measure: it is the measure that makes the SDE driftless. Girsanov's theorem is a useful tool for changing the numeraire of a stochastic process, thereby changing the measure of a stochastic process or SDE.
This section provides an overview of the most important concepts and of notation used in this thesis. Detailed exposition can be found in the standard text books on financial derivatives or interest-rate modelling that are mentioned at the beginning of chapter 2.
The most general definition of a stochastic process is given by:

Definition A.1 (Stochastic process). Let $I \subset \mathbb{R}$ and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A *stochastic process* is a mapping

$$X : I \times \Omega \to \mathbb{R},$$

where, for fixed $t \in I$, $X_t : \omega \mapsto X(t, \omega)$ is a random variable. For fixed $\omega \in \Omega$, $X^\omega : t \mapsto X(t, \omega)$ is a realization of the process, and it is often called a *signal*.

The special case where the stochastic element of a stochastic process is driven by standard normal random variables is called Brownian motion:

Definition A.2 (Brownian motion). A process $W = (W_t : t \geq 0)$ is a $\mathbb{P}$-*Brownian motion* if and only if

(i) W_t is continuous and $W_0 = 0$,

(ii) $W_t \sim N(0,t)$ under $\mathbb{P}$, i.e. the value of W_t is distributed as a standard normal random variable under $\mathbb{P}$,

(iii) it holds that $W_{s+t} - W_s \sim N(0,t)$ under $\mathbb{P}$, and this increment is independent of the history $\mathcal{F}_s$ of what the process did up to time s.

A Brownian motion is also called *Wiener process*. $\mathcal{F}_s$ is the information of W_s up to time s and is called the *filtration* of W_s. W_s is said to be *adapted* to $\mathcal{F}_s$. A process is called $\mathcal{F}$-predictable at time t if it depends only on information up to time t, but not on information at time t itself.

A definition of stochastic processes that is restricted to Brownian motions is given by:

Definition A.3 (Stochastic process, stochastic differential equation). A *stochastic process* X is a continuous process $(X_t : t \geq 0)$ such that X_t can be written as

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds,$$

where the *diffusion coefficient* or *volatility* σ and the *drift* μ are random $\mathcal{F}$-predictable processes such that $\int_0^t (\sigma_s^2 + |\mu_s|) ds < \infty$ for all times t. The differential form of this equation is

$$dX_t = \sigma_t dW_t + \mu_t dt.$$

In the case where σ and μ depend on W_t only through X_t, and where $\sigma(x,t)$ and $\mu(x,t)$ are deterministic functions, the differential is called a *stochastic differential equation (SDE)*:

$$dX_t = \sigma(X_t, t) dW_t + \mu(X_t, t) dt.$$

To obtain an SDE from a stochastic process, Itô's formula can be used:

Theorem A.1 (Itô's formula). *Let X be a stochastic process satisfying $dX_t = \sigma_t dW_t + \mu_t dt$ and f a deterministic, twice continuously differentiable function, then $f(X)$ is also a stochastic process given by*

$$df(X_t) = (\sigma_t f'(X_t))\, dW_t + \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right) dt. \tag{A.1}$$

To devise a trading strategy on a claim X (e.g. an option) based on an underlying S that is modelled by a stochastic process, the concept of martingales is required:

Definition A.4 (Martingale). A stochastic process M_t is a martingale with respect to a probability measure $\mathbb{P}$ if and only if

(i) $\mathbb{E}_{\mathbb{P}}(|M_t|) < \infty$, $\forall t$,

(ii) $\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = M_s$, $\forall s \leq t$.

A martingale measure is a measure which makes the expected future value, conditioned on its history, its present value. It is not expected to drift upwards or downwards.
It may be that one is faced with a stochastic process that is not a martingale under a given probability measure $\mathbb{P}$. In this case, it is possible to express one measure $\mathbb{P}$ through another measure $\mathbb{Q}$, given that the measures are *equivalent*, meaning that they operate on the same sample space Ω and that for any event $A \in \Omega$ it holds that

$$\mathbb{P}(A) > 0 \iff \mathbb{Q}(A) > 0.$$

The transfer from one measure $\mathbb{P}$ to another measure $\mathbb{Q}$ is achieved through the *Radon-Nikodym derivative*:

Definition A.5 (Radon-Nikodym derivative). Given two equivalent measures $\mathbb{P}$ and $\mathbb{Q}$, $\mathbb{P}$ can be expressed through $\mathbb{Q}$ with the random variable $\frac{d\mathbb{P}}{d\mathbb{Q}}$, called the Radon-Nikodym derivative. The expectation of X with respect to $\mathbb{P}$ is

$$\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}X\right).$$

Furthermore, a process on the filtration $\mathcal{F}_T$ using the Radon-Nikodym derivative can be defined:

$$\zeta_t = \mathbb{E}_{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t\right).$$

This process is also called the Radon-Nikodym derivative as it is the transfer of the Radon-Nikodym derivative for random variables to stochastic processes. The Radon-Nikodym derivative ζ_t satisfies the following property:

$$E_{\mathbb{Q}}(X_t|\mathcal{F}_s) = \zeta_s^{-1} E_{\mathbb{P}}(\zeta_t X_t|\mathcal{F}_s), \quad s \leq t \leq T.$$

Applying the Radon-Nikodym derivative to a $\mathbb{P}$-Brownian motion changes the drift of the underlying process, but has no effect on the volatility of the process.
Mapping an SDE under $\mathbb{P}$ to an SDE under $\mathbb{Q}$ is formalized by Girsanov's theorem:

Theorem A.2 (Girsanov theorem, Cameron-Martin-Girsanov theorem). *If W is a $\mathbb{P}$-Brownian motion and γ is an $\mathcal{F}$-predictable process satisfying the boundedness condition $\mathbb{E}_{\mathbb{P}}(exp(\frac{1}{2}\int_0^T \gamma_t^2 dt) < \infty$, then there exists a measure $\mathbb{Q}$ such that*

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$

(ii) $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$

(iii) $\tilde{W}_t = W_t + \int_0^T \gamma_s ds$ is a $\mathbb{Q}$-Brownian motion.

Given an SDE $dX_t = \sigma_t dW_t + \mu_t d_t$, to obtain an SDE with drift ν_t requires applying the Girsanov theorem with $\gamma_t = (\mu_t - \nu_t)/\sigma_t$.
An application of the Radon-Nikodym derivative and Girsanov's theorem is the choice of *numeraire* for an asset. A numeraire may be any positive, non-dividend paying, tradeable asset. It is the unit in which other asset prices are expressed. The most natural numeraire is the bank account given by equation (2.1). A popular numeraire in interest rate modelling is the T-maturity bond with price $P(t,T)$ at time t. The martingale measure for this bond is called the T-forward measure $\mathbb{P}_T$, and makes the forward rate $f(t,T)$ as well as the LIBOR rate for borrowing up until time T $\mathbb{P}_T$-martingales.
The choice of numeraire of stochastic processes provides an important tool in evaluating financial instruments under the assumption of arbitrage-free markets.

Appendix B

Derivation of full-rank, two-parameters parameterization formula

The stable, full-rank, two-parameters parameterization for instantaneous correlation introduced by (Schoenmakers and Coffey, 2000) (see equation (4.5)) is referenced in (Brigo, 2002). The reference, however, is not correct. This appendix derives the formula to show that the original formula is indeed correct.
The formula given by (Schoenmakers and Coffey, 2000) is

$$\rho_{i,j} = \exp\left(-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{M-i-j+1}{M-2}\right)\right),$$

where $0 \leq \eta \leq -\ln\rho_\infty, \quad i,j = 1,\ldots,m$. The general form of the correlation structure is given by

$$\rho_{i,j} = \frac{c_i}{c_j} = \exp\left(-\sum_{l=i+1}^{M}\min(l-i, j-i)\Delta_l\right), \quad i < j,$$

where c_i is given by equation (4.4). Setting $\Delta = \Delta_2 = \cdots = \Delta_{M-1}$ yields

$$\begin{aligned}
\rho_{i,j} &= \exp\left(-\left(\Delta\sum_{l=1}^{j-i} l\right) - \left(\Delta\sum_{l=j+1-i}^{M-1-i} j-i\right) - \Delta_M(j-i)\right) \\
&= \exp\left(-\Delta\frac{(j-i)(j+1-i)}{2} - \Delta(j-i)(M-1-j) - \Delta_M(j-i)\right) \\
&= \exp\left(-\Delta(j-i)\left(M + \frac{j+1-i-2-2j}{2}\right) - \Delta_M(j-i)\right) \\
&= \exp\left(-\Delta(j-i)\left(M - \frac{j+i+1}{2}\right) - \Delta_M(j-i)\right)
\end{aligned}$$

Introducing new parameters

$$\begin{aligned}
\rho_\infty &:= \rho_{1,M} = \exp\left(-(M-1)\left(\frac{M-2}{2}\Delta + \Delta_M\right)\right), \\
\eta &:= \frac{\Delta(M-1)(M-2)}{2},
\end{aligned}$$

Δ and Δ_M become

$$\begin{aligned}
\Delta &= \frac{2\eta}{(M-1)(M-2)} \\
\Delta_M &= \frac{-\ln\rho_\infty}{(M-1)} - \frac{M-2}{2}\Delta \\
&= \frac{-\ln\rho_\infty}{(M-1)} - \frac{\eta}{(M-1)}.
\end{aligned}$$

$\rho_{i,j}$ can then be expressed as

$$\begin{aligned}
\rho_{i,j} &= \exp\left(-|i-j|\left(\frac{2\eta}{(M-1)(M-2)}\cdot\frac{2M-j-i-1}{2} - \frac{\ln\rho_\infty+\eta}{(M-1)}\right)\right) \\
&= \exp\left(-\frac{|i-j|}{M-1}\left(\eta\frac{2M-j-i-1}{M-2} - \eta - \ln\rho_\infty\right)\right) \\
&= \exp\left(-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{2M-M+2-j-i-1}{M-2}\right)\right) \\
&= \exp\left(-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{M-j-i+1}{M-2}\right)\right).
\end{aligned}$$

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